MAT-2

* $1-1+1-1+1-1+---+(-1)^n+---$ Alterne seri

n'in tekliğinde ve çiftliğinde limit farklıdır, tek bir limit değeri yoktur. Bu yüzden seri ıraksaktır.

$\lim\limits_{n\to\infty}$ ∄

<u>Teorem:1-)</u> Bir serinin baş tarafına sonlu sayıda herhangi terimler ilave etmek veya baş tarafından sonlu sayıda terimler atmakla serinin karakteri değişmez.
yak. yak. veya ırak. ırak. ise

<u>Teorem:2-)</u> a) $\sum\limits_{n=1}^{\infty} u_n$, $\sum\limits_{n=1}^{\infty} v_n$ serisi ve k skaler bir sayı olmak üzere u_n ve v_n yakınsak seriler ise $\sum\limits_{n=1}^{\infty} k.u_n$ ve $\sum\limits_{n=1}^{\infty} k.v_n$ de yakınsaktır.

$\sum\limits_{n=1}^{\infty} k.u_n = k\sum\limits_{n=1}^{\infty} u_n$ Bir skalerle çarpılan seri karakterini değiştirmez.

b) $\sum\limits_{n=1}^{\infty} (u_n \mp v_n) = \sum\limits_{n=1}^{\infty} u_n \mp \sum\limits_{n=1}^{\infty} v_n$

u_n ve v_n yakınsak ise bu şartlarda elde edilen seriler de yakınsaktır, ıraksaksa ıraksaktır.

<u>Yakınsaklık İçin Gerekli Şart:</u>

<u>Teorem:</u> Yakınsak bir seride $\lim\limits_{n\to\infty} u_n = 0$'dır.

Yakınsak her dizinin genel terimi 0'a gider. Bu gerek şarttır, yeter şart değildir.

$$\underbrace{\overbrace{u_1+u_2+u_3+---+u_{n-1}}^{S_{n-1}}+u_n}_{S_n}$$

$S_n = S_{n-1} + u_n$

$S_n - S_{n-1} = u_n$

$$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} u_n$$

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} u_n$$

$$\underbrace{S - S}_{0} = \lim_{n\to\infty} u_n$$

Genel teriminin limiti 0'a gitmeyen bir seri için yakınsaktır diyemeyiz.

$\cdot\ u_n = \dfrac{1}{\sqrt{n}}$

Bir serinin genel teriminin 0'a gitmesi o serinin yakınsak olduğu anlamına gelmez.

Örnek: $u_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} n^{\frac{1}{n}}} = \frac{1}{e^0} = 1$$

$y = n^{\frac{1}{n}} = \infty^0$

$\ln y = \frac{1}{n} \ln n$

$\ln y = \frac{\ln n}{n}$

$$\lim_{n\to\infty} (\ln y) = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= \lim_{n\to\infty} \frac{1/n}{1}$$

$$\lim_{n\to\infty} (\ln y) = 0$$

$$\lim_{n\to\infty} y = 1$$

Genel teriminin limiti 0'a gitmediği için bundan elde edilen seri ıraksaktır.

* $\sum a.r^n = \dfrac{1-r^n}{1-r}$

* $1+2+3+ \cdots +n = \dfrac{n(n+1)}{2}$

* $1+3+5+ \cdots +2n-1 = n^2$

Örnek: $u_n = \dfrac{1}{n(n+1)(n+2)}$ olan serinin yakınsaklığı veya ıraksaklığı halinde ne söyleyebilirsiniz?

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots$$

$$\frac{1}{n(n+1)(n+2)} = \frac{A^{=1/2}}{n} + \frac{B^{=-1}}{n+1} + \frac{C^{=1/2}}{n+2}$$

$$1 \equiv A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)$$

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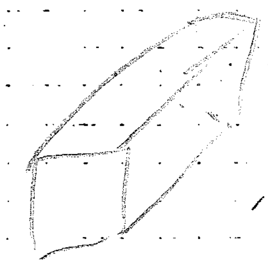
$\boxed{A = 1/2}$
$\boxed{B = -1}$
$\boxed{C = 1/2}$

$$u_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4}$$

$$\begin{aligned} S_n = & \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{6}} \\ & + \tfrac{1}{4} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{8}} \\ & + \cancel{\tfrac{1}{6}} - \tfrac{1}{4} + \tfrac{1}{10} \\ & + \cancel{\tfrac{1}{8}} - \tfrac{1}{5} + \tfrac{1}{12} \\ & + \\ & \vdots \\ & + \tfrac{1}{2n-2} - \tfrac{1}{n} + \tfrac{1}{2n+2} \\ & + \cancel{\tfrac{1}{2n}} - \tfrac{1}{n+1} + \tfrac{1}{2n+4} \end{aligned}$$



$$S_n = \frac{1}{4} + \frac{1}{2n+2} - \frac{1}{n+1} + \frac{1}{2n+4}$$

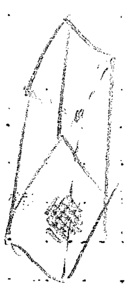
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2n+2} - \frac{1}{n+1} + \frac{1}{2n+4}\right)$$

$S = \frac{1}{4}$ Toplam bulunabildiği için bu seri yakınsaktır.

<u>Pozitif Sonsuz Terimli Serilerin İntegral Yardımı İle Yakınsaklık Tayini (İntegral Kuralı):</u>

$$S_n = u_1 + u_2 + \dots + u_n = \sum_{n=1} u_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{\infty} u_n = \int_1^{\infty} u_n\,dn$$



<u>Teorem:</u> 1.'den itibaren her terimi kendisinden önce gelenden daha küçük ve genel terimi $f(n)$ olan pozitif terimli bir seride $n \geq 1$ için $f(n) > 0$, azalan ve sürekli ise $\int_1^{\infty} f(n)dn$ integrali mevcut olduğu zaman seri yakınsak olmadığı zaman seri ıraksaktır.

Örnek: $u_n = \frac{1}{n}$ olan sonsuz terimli serinin karakterini inceleyin.

$n \geq 1$

$$\int_1^{\infty} \frac{1}{n} dn = \lim_{A \to \infty} \int_1^{A} \frac{1}{n} dn$$
$$= \lim_{A \to \infty} \left| \ln n \right|_1^A$$
$$= \lim_{A \to \infty} (\ln A - \ln 1)$$
$$= \ln \infty - \ln 1$$
$$= \infty \quad \text{Seri ıraksaktır.}$$

$\cancel{A}$ $\int \frac{1}{n^p}$ $p=1$ ıraksak.

Örnek: $$\int_1^{\infty} \frac{1}{n^p} dn = \lim_{A \to \infty} \int_1^{A} n^{-p} dn$$
$$= \lim_{A \to \infty} \left| \frac{n^{-p+1}}{1-p} \right|_1^A$$
$$= \lim_{A \to \infty} \left(\frac{A^{-p+1}}{1-p} - \frac{1}{1-p} \right)$$
$$= \lim_{A \to \infty} \left(\frac{1}{1-p} \cdot \frac{1}{A^{p-1}} - \frac{1}{1-p} \right)$$

$\begin{cases} p > 1 \text{ ise sonuç sonlu olduğu için seri yakınsak} \\ p < 1 \text{ ise ıraksak} \end{cases}$

$\cancel{A}$ $\boxed{\frac{1}{n^p}}$ $p=1$ ise ıraksak
$p>1$ ise yakınsak
$p<1$ ise ıraksak

Mukayese Kuralı:

Elemanları $a_1 + a_2 + a_3 + \dots + a_n$ olan pozitif terimli bir seri ve elemanları $u_1 + u_2 + u_3 + \dots + u_n$ seriler verilmiş olsun. Eğer a_n serisi yakınsak ve u_n serisindeki her terim, a_n serisindeki karşılığı olan terimden küçük (büyük) veya ona eşitse u_n serisi de yakınsaktır (ıraksaktır). Tersi için de geçerlidir. (2. terimden itibaren)

Örnek 1-) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Bu serinin karakterini tayin ediniz.

$\frac{1}{n^2}$ yakınsak bir seridir.

$a_n = \frac{1}{n^2}$ Yak.

$u_n = \frac{1}{n(n+1)}$ Yak.

$\frac{1}{n(n+1)} < \frac{1}{n^2}$

$n^2 < n(n+1)$

Örnek 2-) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$u_n = \frac{1}{n^n}$

$a_n = \left(\frac{1}{2^n}\right)^n$

$n^n > 2^n$

Yak. olur. $\boxed{u_n < a_n \text{ Yak.}}$

$\frac{1}{n^n} < \frac{1}{2^n}$

Örnek 3-) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}$ karakteri nedir?

$a_n = \frac{1}{n^4}$

$(2n+1)^4 > n^4$

$\frac{1}{(2n+1)^4} < \frac{1}{n^4}$

Yak.

Örnek 4-) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$u_n = \frac{1}{\sqrt{n}}$

$a_n = \frac{1}{n}$

$n > \sqrt{n}$

$\frac{1}{n} < \frac{1}{\sqrt{n}}$ Iraksak

Örnek 5-) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

$u_n = \frac{1}{2n+1}$

$\boxed{a_n = \frac{1}{n}}$ Iraksak

$$2n+2 > 2n+1$$
$$\frac{1}{2n+2} < \frac{1}{2n+1}$$
Iraksak

$$n-1 < n$$
$$\frac{1}{n-1} > \frac{1}{n}$$

D' Alembert Kuralı:

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = R \quad \begin{cases} R<1 & \text{ise yakınsak} \\ R>1 & \text{ise ıraksak} \\ R=1 & \text{ise şüpheli hal. Raabe kriterine bak.} \end{cases}$$

Yakınsak bir serinin terimleri gittikçe küçülür.

$$\frac{u_{n+1}}{u_n} < \frac{u_n}{u_n}$$
$$\frac{u_{n+1}}{u_n} < 1$$

Iraksak serilerin terimleri gittikçe büyür.

Örnek: $u_n = \frac{n}{2^n}$

$$u_{n+1} = \frac{n+1}{2^{n+1}}$$

$$\lim_{n\to\infty}\left(\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right) = \frac{1}{2}$$ Seri yakınsaktır.

Örnek: $u_n = \frac{n!}{10^n}$

$$u_{n+1} = \frac{(n+1)!}{10^{n+1}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}\right]$$

$$= \lim_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!}$$

4. soruyu yap.

$$= \infty$$ Iraksak

Cauchy Kuralı:

Pozitif sonsuz terimli bir seri verilmiş olsun. 9

$u_1+u_2+u_3+\dots+u_n+\dots$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = R \begin{cases} R<1 & \text{Yakınsak} \\ R>1 & \text{Iraksak} \\ R=1 & \text{Birşey söylenemez.} \end{cases}$$

Örnek: $u_n=(\log_e n)^n$

$$\ln x = \frac{\log x}{\log e}$$

$$\lim_{n\to\infty} \sqrt[n]{(\log_e n)^n} = \lim_{n\to\infty} \ln n = \infty \quad \text{Iraksak}$$

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\log_e 10}$$

Örnek: $u_n = \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 \quad \text{Yakınsak}$$

Örnek: $\sum_{n=1}^{\infty} (\sqrt[n]{n}-1)^n$

$$\lim_{n\to\infty} \sqrt[n]{(\sqrt[n]{n}-1)^n} = \lim_{n\to\infty} (\sqrt[n]{n}-1)$$

$$\lim_{n\to\infty} (n^{1/n}) - \lim_{n\to\infty} 1 = 0 \quad \text{Yakınsak}$$

$$\frac{-8+4}{16+8} \quad \frac{-4}{24} \quad -\frac{1}{6}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln n$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty} \frac{1/n}{1} = 0$$

$$\lim_{n\to\infty} (n^{1/n}) = e^0 = 1$$

$$\frac{-2+1}{3}$$

Örnek: $\sum_{n=1}^{\infty} 3^{\frac{-2n^2+n}{n+2}}$

$$\lim_{n\to\infty} \sqrt[n]{3^{\frac{-2n^2+n}{n+2}}} = \lim_{n\to\infty} \left(3^{\frac{-2n^2+n}{n+2}}\right)^{1/n} = \lim_{n\to\infty} 3^{\frac{-2n^2+n}{n^2+2n}} = 3^{-2} = \frac{1}{9} \quad \text{Yakınsak}$$

$$* \left.\begin{matrix} \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = R_1 \\ \lim_{n\to\infty} \sqrt[n]{u_n} = R_2 \end{matrix}\right\} R_1 = R_2$$

Hem D'Alembert hem de Cauchy ile çözülen sorularda.

Oran Kuralı:

$\frac{u_n}{v_n}$

Bir nevi mukayese etmedir.

$\lim_{n\to\infty} \frac{u_n}{v_n} = l$ ise $\begin{cases} l\neq 0 & \text{ise seriler aynı karakterdedir.} \\ l=0 & \text{ise } v_n \text{ yakınsaksa } u_n \text{ de yakınsaktı.} \\ l\to\infty & \text{ise } v_n \text{ ıraksaksa } u_n \text{ de ıraksaktı.} \end{cases}$

Örnek: $u_n = \frac{a.n^p + b.n^{p-1} + c.n^{p-2} + \cdots}{a_1.n^q + b_1.n^{q-1} + c_1.n^{q-2} + \cdots}$

$v_n = \frac{n^p}{n^q}$

$$\lim_{n\to\infty}\left(\frac{an^p+bn^{p-1}+\cdots}{a_1n^q+b_1n^{q-1}+\cdots}\cdot\frac{n^q}{n^p}\right) = \lim_{n\to\infty}\frac{\cancel{n^p}\left(a+b.\frac{1}{n}+c.\frac{1}{n^2}+\cdots\right)\cancel{n^q}}{\cancel{n^q}\left(a_1+b_1.\frac{1}{n}+c_1.\frac{1}{n^2}+\cdots\right)\cancel{n^p}}$$

$= \frac{a}{a_1}$ v_n ve u_n aynı karakterde

$v_n = \frac{1}{n^{q-p}}$ $\begin{cases} q-p\leq 1 & \text{ıraksak} \\ q-p>1 & \text{yakınsak} \end{cases}$

Örnek: $u_n = \frac{\sqrt[3]{n^4-16}}{n^2\sqrt{n+1}}$

$v_n = \frac{\sqrt[3]{n^4}}{n^2\sqrt{n}} = \frac{n^{4/3}}{n^2.n^{1/2}}$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^4-16}}{\cancel{n^2}\sqrt{n+1}}\cdot\frac{\cancel{n^2}\sqrt{n}}{\sqrt[3]{n^4}} = \lim_{n\to\infty}\frac{\cancel{n^{4/3}}\sqrt[3]{1-\frac{16}{n^4}}}{\cancel{n^{1/2}}\sqrt{1+\frac{1}{n}}}\cdot\frac{\cancel{n^{1/2}}}{\cancel{n^{4/3}}} = 1$$ u_n ve v_n aynı karakterde

$v_n = \frac{1}{n^{2+\frac{1}{2}-\frac{4}{3}}} = \frac{1}{n^{7/6}}$ Yakınsak u_n de yakınsaktır.

Örnek: $u_n = \frac{1}{\sqrt{n}}\sin\frac{\pi}{n}$ Serisinin karakterini belirtiniz.

$v_n = \frac{1}{\sqrt{n}}\cdot\frac{1}{n} = \frac{1}{n^{3/2}}$ Yakınsak

$$\lim_{n\to\infty}\frac{\cancel{\frac{1}{\sqrt{n}}}\sin\frac{\pi}{n}}{\cancel{\frac{1}{\sqrt{n}}}\cdot\frac{1}{n}} = \pi$$ u_n de yakınsaktır.

$\begin{cases} \lim_{x\to 0}\frac{\sin\pi x}{x} = \pi \\ \lim_{x\to\infty}\frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \pi \end{cases}$

Raabe Kuralı:

D'Alembert kuralı uygulandığında $R=1$ çıkıyorsa uygulanır.

$$\lim_{n\to\infty}\left[\frac{u_{n+1}}{u_n}-1\right]n=\ell \begin{cases} \ell<-1 & \text{Yakınsak} \\ \ell>-1 & \text{Iraksak} \end{cases}$$

<u>Örnek:</u> $u_n=\dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$ Serinin karakterini belirtiniz.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\cancel{1.3.5\ldots(2n-1)}(2n+1)}{\cancel{2.4.6\ldots 2n}(2n+2)}\cdot\frac{\cancel{2.4.6\ldots 2n}}{\cancel{1.3.5\ldots(2n-1)}}=1$$

$$\lim_{n\to\infty}\left[\frac{2n+1}{2n+2}-1\right]n=\lim_{n\to\infty}\left(\frac{2n+1-2n-2}{2n+2}\right)n$$

$$=\lim_{n\to\infty}\left(\frac{-n}{2n+2}\right)$$

$$=-\frac{1}{2}>-1 \quad \text{Iraksak}$$

ALTERNATİF SERİLER

Bu tip serilerde terimler $+,-,+,-,\ldots$ şeklinde gider.

<u>Teorem:</u>

$u_1-u_2+u_3-u_4+\ldots+(-1)^n u_n$ Böyle bir alterne seride n artarken serinin terimleri mutlak değer bakımından azalıyor veya sabit kalıyor ve $\lim_{n\to\infty} u_n=0$ oluyorsa seri yakınsaktır.

Alterne seri olarak yakınsak ama pozitif terimli olarak ıraksak

<u>Örnek:</u> $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\dfrac{1}{6}+\ldots+(-1)^n\dfrac{1}{n}+\ldots$

$\lim_{n\to\infty}(u_n)=\lim_{n\to\infty}\dfrac{1}{n}=0$ Alterne seri olarak yakınsaktır.

$|u_1|+|u_2|+|u_3|+\ldots+|u_n|+\ldots$ Pozitif terimli olarak ıraksaktır.

$1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}+\ldots$

Bu seriye şartlı yakınsak denir.

*Bir seri alterne seri olarak yakınsak, pozitif terimli olarak ıraksaksa bu seri şartlı yakınsaktır.

Örnek: $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots+(-1)^{n+1}\frac{1}{n^2}+\dots$

$\lim_{n\to\infty}(u_n)=\lim_{n\to\infty}\frac{1}{n^2}=0$ Alterne seri olarak yakınsaktır.

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots$ Pozitif terimli olarak yakınsaktır.

* Alterne bir seri, alterne seri olarak yakınsak, pozitif terimli olarak da yakınsaksa mutlak yakınsaktır.

Örnek: $\frac{1}{2}-\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+\dots+(-1)^{n+1}\frac{n}{2^n}+\dots$ serinin karakterini inceleyiniz.

$\lim_{n\to\infty}(u_n)=\lim_{n\to\infty}\left(\frac{n}{2^n}\right)=\lim_{n\to\infty}\left(\frac{1}{2^n \ln 2}\right)=0$ Alterne seri olarak yakınsak

$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+\dots+\frac{n}{2^n}$

D'Alembert'e göre;
$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=\lim_{n\to\infty}\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}=\frac{1}{2}$ Yakınsak.

Seri mutlak yakınsaktır.

DEĞİŞKEN TERİMLİ SERİLER (KUVVET SERİLERİ)

$\sum_{n=0}^{\infty}C_n x^n$ şeklinde tanımlanmış olan bir seridir.

$$\sum_{n=0}^{\infty}C_n x^n=C_0x^0+C_1x^1+C_2x^2+\dots+C_nx^n+\dots$$

$$\sum_{n=0}^{\infty}C_n(x-a)^n=C_0(x-a)^0+C_1(x-a)^1+\dots+C_n(x-a)^n+\dots$$

$u_n=C_nx^n$ serinin karakterini tayin ediniz.

$u_{n+1}=C_{n+1}x^{n+1}$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$ Yakınsak

$\lim_{n\to\infty}\left|\frac{C_{n+1}x^{n+1}}{C_nx^n}\right|=\lim_{n\to\infty}\frac{C_{n+1}}{C_n}|x|=R|x|$

$R|x|<1$ ise Yakınsaktır.

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$|x| < \frac{1}{R}$

$\boxed{-\frac{1}{R} < x < \frac{1}{R}}$ Verilen serinin yakınsaklık yarıçapı

Örnek: $\sum_{n=1}^{\infty} \frac{x^n}{n}$ serisinin yakınsaklık yarıçapını bulunuz (x'in hangi değerleri için seri yakınsak)

$u_n = \frac{x^n}{n}$

$u_{n+1} = \frac{x^{n+1}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| < 1$$

$|x| < 1$

$\boxed{-1 \leq x < 1}$

eşit koyabiliyoruz

$x=-1$ için $u_n = \frac{(-1)^n}{n}$ $\quad -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{(-1)^n}{n} + \cdots$

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0$$

şartlı yakınsak

$x=1$ için $u_n = \frac{1^n}{n} = \frac{1}{n}$ $\quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

Iraksak

Örnek: $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{2^{n+1} \cdot (n+1)^2} \cdot \frac{2^n \cdot n^2}{(x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2+2n+1} \cdot \frac{1}{2}(x+1) \right|$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{2}|x+1| < 1$$

$|x+1| < 2$

$-2 < x+1 < 2$

$-3 \leq x \leq 1$

$x=-3$ için $u_n = \frac{(-1)^n(-2)^n}{2^n \cdot n^2} = \frac{2^n}{2^n n^2} = \frac{1}{n^2}$ Yakınsak $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$x=1$ için $u_n = \frac{(-1)^n \cancel{2^n}}{\cancel{2^n} \cdot n^2} = \frac{(-1)^n}{n^2}$ Mutlak yakınsaktır.

Taylor ve Maclorea Serisiyle Yaklaşık Değer Hesabı:

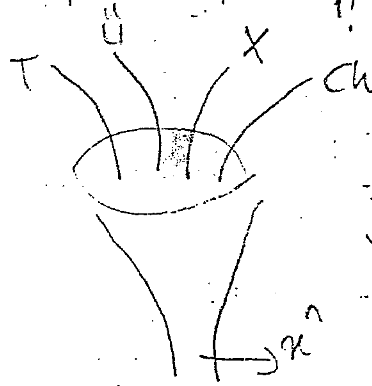
$f(x)$ fonksiyonunun Taylor serisine açılımı;

$$f(x) = f(a) + \frac{1}{1!}(x-a)^1 f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \dots$$

$x-a=h$ pozitif küçük bizce seçilebilecek bir değer olmak üzere $x=a+h$

yukarıdaki açılımda x yerine $a+h$ yazılırsa;

$$f(a+h) = f(a) + \frac{1}{1!}(h) f'(a) + \frac{1}{2!} h^2 f''(a) + \dots + \frac{1}{n!}(h)^n f^{(n)}(a) + \dots$$



Taylor ve Maclora serisi birbirinden farklı fonsiyonları sonuç olarak polinomiyel şekilde verir.

Örnek $\sin 61° = ?$

$f(x) = y = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$$\sin(60°+1°) = \frac{\sqrt{3}}{2} + \frac{1}{1!}\left(\frac{\pi}{180}\right) \cdot \frac{1}{2} + \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 \left(-\frac{\sqrt{3}}{2}\right) + \dots$$

Ne kadar çok terim yazarsak hata payı o kadar düşük olur.

Bundan yararlanarak yaklaşık hesap yapılabilir.

Örnek: $\sqrt[3]{1,2} = ?$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\qquad f'(1) = \frac{1}{3}$

$f''(x) = -\frac{1}{3} \cdot \frac{2}{3} x^{-5/3}$ $\qquad f''(1) = -\frac{2}{9}$

$f'''(x) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot x^{-8/3}$ $\qquad f'''(1) = \frac{10}{27}$

$\sqrt[3]{1,2} = \sqrt[3]{1 + \frac{2}{10}}$

$$\sqrt[3]{12} = 1+\frac{1}{1!}\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)+\frac{1}{2!}\left(\frac{1}{5}\right)^2\left(-\frac{2}{9}\right)+\frac{1}{3!}\left(\frac{1}{5}\right)^3\left(\frac{10}{27}\right)+\cdots$$

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Binom Açılımı:

$f(x)=(a+x)^m$ $\qquad f(0)=a^m$

$f'(x)=m.(a+x)^{m-1}$ $\qquad f'(0)=m.a^{m-1}$

$f''(x)=m(m-1)(a+x)^{m-2}$ $\qquad f''(0)=m(m-1)a^{m-2}$

$f'''(x)=m(m-1)(m-2)(a+x)^{m-3}$ $\qquad f'''(0)=m(m-1)(m-2)a^{m-3}$

Maclorene göre açalım:

$$(a+x)^m = a^m+\frac{1}{1!}x.m.a^{m-1}+\frac{1}{2!}x^2.m(m-1)a^{m-2}+\frac{1}{3!}x^3m(m-1)(m-2)a^{m-3}+\cdots$$

$$= a^m+\frac{m}{1!}a^{m-1}.x+\frac{m(m-1)}{2!}a^{m-2}.x^2+\frac{m(m-1)(m-2)}{3!}a^{m-3}x^3+\cdots+\frac{m(m-1)(m-2)\ldots(m-r+2)}{(r-1)!}a^{m-r+1}x^{r-1}$$

Örnek: $\sqrt{1+x}=?$ verilen ifadeyi seriye açınız.

$$\sqrt{1+x}=(1+x)^{1/2}$$

$$(1+x)^{1/2}=1^{1/2}+\frac{1/2}{1!}1^{m-1}.x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}1^{m-2}.x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}1^{\frac{1}{2}-3}x^3+\cdots$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\cdots$$

SERİLERLE İŞLEMLER

Serilerin Toplanması ve Çıkarılması:

$P(x)=a_0+a_1x^1+a_2x^2+a_3x^3+\cdots+a_nx^n+\cdots$ iki. ar.

$Q(x)=b_0+b_1x^1+b_2x^2+b_3x^3+\cdots+b_nx^n+\cdots$ / için de geçerli

$$P(x)+Q(x)=(a_0+b_0)+(a_1+b_1)x^1+(a_2+b_2)x^2+\cdots+(a_n+b_n)x^n+\cdots$$

$P(x)+Q(x)$ polinomunun yakınsaklık aralığı, sonucu meydana getiren $P(x)$ ve $Q(x)$ polinomlarının ortak yakınsaklık aralığıdır.

Ör/ $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$

} toplam serinin yakınsaklık aralığını bulunuz.

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+---+(-1)^{n+1}\frac{x^n}{n}+---$$

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$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+---+\frac{x^n}{n}$$

$$+\overline{\quad 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+---\right)}$$

I. serinin yak. aralığı $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|=|x|<1$ $\boxed{-1<x\leq 1}$

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots-\frac{1}{n}$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+---+\frac{1}{n}\right) \text{ Iraksak}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+---(-1)\frac{1}{n} \text{ Şartlı Yakınsak}$$

II. serinin yak. aralığı $\boxed{-1\leq x<1}$

-1 için $-1+\frac{1}{2}-\frac{1}{3}+---$ Şartlı Yakınsak

1 için Iraksak

Toplam serinin yakınsaklık aralığı $-1<x<1$

<u>Örnek:</u> $f(x)=\frac{1}{2}(e^x-e^{-x})$ Seriye açınız.

$$e^x=1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+---+\frac{1}{n!}x^n+---$$

$$e^{-x}=\mp 1\pm\frac{1}{1!}x\mp\frac{1}{2!}x^2\pm\frac{1}{3!}x^3\mp---+\frac{(-1)^{n+1}}{n!}x^n+---$$

$$-$$

$$e^x-e^{-x}=2\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+---+\frac{x^{2n-1}}{(2n-1)!}+---\right)$$

$$f(x)=\frac{1}{2}(e^x-e^{-x})=\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+---+\frac{x^{2n-1}}{(2n-1)!}+---\right)$$

Serilerin Çarpılması: 17

Çarpıma giren serilerin ortak yakınsaklık aralığı, sonucun yakınsaklık aralığıdır.

Örnek: $f(x) = \frac{1}{1-x^2}$

$$f(x) = \frac{1}{1-x^2} = \left(\frac{1}{1+x}\right) \cdot \left(\frac{1}{1-x}\right) = P(x) \cdot Q(x)$$

$f(x) = \frac{1}{1+x}$ $\quad f(0) = 1$

$f'(x) = \frac{-1}{(1+x)^2}$ $\quad f'(0) = -1!$

$f''(x) = \frac{2}{(1+x)^3}$ $\quad f''(0) = 2!$

$f'''(x) = \frac{-6}{(1+x)^4}$ $\quad f'''(0) = -3!$

$f^{IV}(x) = \frac{+24}{(1+x)^5}$ $\quad f^{IV}(0) = 4!$

$$\frac{1}{1+x} = 1 + \frac{1}{1!}x(-1) + \frac{1}{2!}x^2(2!) + \frac{1}{3!}x^3(-3!) + \frac{1}{4!}x^4(4!) + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n$$

$$\begin{array}{r|l} 1 & 1+x \\ \hline -1 \mp x & 1-x+x^2-x^3 \\ -x & \\ -x-x^2 & \\ x^2 & \\ x^2+x^3 & \\ -x^3 & \end{array}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$$

$$-1 < x < 1$$

-1 için $1+1+1---$ ırak.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x} \cdot \frac{1}{1-x} = (1 - x + x^2 - x^3 + x^4 - \cdots)(1 + x + x^2 + x^3 + x^4 + \cdots)$$

$$= 1 + (1-1)x + (1-1+1)x^2 + (1-1+1-1)x^3 + (1+1-1-1+1)x^4 + \cdots$$

$$= 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots$$

Örnek: $f(x) = e^x \cdot \cos x$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

x

$$e^x \cdot \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots$$

Serilerin Bölümü:

Örnek: $f(x)=tg\,x=\dfrac{\sin x}{\cos x}$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+---$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+---$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+--- \quad \Big| \quad 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+---$$

$$x+\frac{x^3}{3}+\frac{2x^5}{15}+---$$

$$tg\,x=x+\frac{x^3}{3}+\frac{2x^5}{15}+---$$

SERİLERİN TÜRETİLMESİ

$f'(x)$ esas fonksiyonun açılımının türevidir.

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+---$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+---$$

$$\frac{1}{(1-x)}=1+x+x^2+x^3+---+x^n+---$$

türevi

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+---+---$$

$$\begin{array}{r|l} 1 & 1-x \\ \hline -1+x & 1+x+x^2+--- \\ x & \\ -x+x^2 & \\ \hline x^2 & \\ -x^2+x^3 & \\ \hline x^3 & \end{array}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|<1$$

$$|x|<1$$
$$-1<x<1$$

Orjinalinin yakınsaklık aralığı türevinin de yakınsaklık aralığıdır. Fakat orjinali için tanımlanan sınır değerleri türevi için dahil olmayabilir.

SERİLERİN İNTEGRASYONU

$$\int \frac{dx}{1+x} = \int \left(1-x+x^2-x^3+x^4-x^5+\cdots\right)dx$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+C$$

$$x=0 \Rightarrow \ln 1 = C = 0$$

$$\begin{array}{r|l} 1 & 1+x \\ \hline -1\mp x & 1-x+x^2-\cdots \\ -x & \\ \mp x \mp x^2 & \\ \hline x^2 & \\ -x^2+x^3 & \\ \hline \end{array}$$

Örnek: $$\int \frac{\sin x}{x}dx = \int \frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)dx$$

$$= \int \left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\right)dx$$

Örnek: $$\int e^{x^2}dx = \int \left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots\right)dx$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$e^{x^2} = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right)$$ Belirsizlik ortadan kalktı.

$$=1$$

Örnek: $$\lim_{x\to 0}\frac{\sqrt[3]{x+27}-3}{\sqrt[4]{x+16}-2} = ? \quad \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(x+27)^{1/3}-3}{(x+16)^{1/4}-2} = \lim_{x\to 0}\frac{\left[\cancel{27^{1/3}}+\frac{1/3}{1!}\cdot 27^{\frac{1}{3}-1}\cdot x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot 27^{\frac{1}{3}-2}\cdot x^2+\cdots\right]-\cancel{3}}{\left[\cancel{16^{1/4}}+\frac{1/4}{1!}\cdot 16^{\frac{1}{4}-1}\cdot x+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\cdot 16^{\frac{1}{4}-2}\cdot x^2+\cdots\right]-\cancel{2}}$$

$$= \lim_{x\to 0}\frac{\cancel{x}\left[\frac{1}{27}+\beta_1 x+\beta_2 x^2+\cdots\right]}{\cancel{x}\left[\frac{1}{32}+\alpha_1 x+\alpha_2 x^2+\cdots\right]}$$

$$= \frac{1}{27}\cdot 32$$

$$= \frac{32}{27}$$

$$(a+x)^n = a^n+\frac{n}{1!}a^{n-1}\cdot x+\frac{n(n-1)}{2!}a^{n-2}\cdot x^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}\cdot x^3+\cdots$$

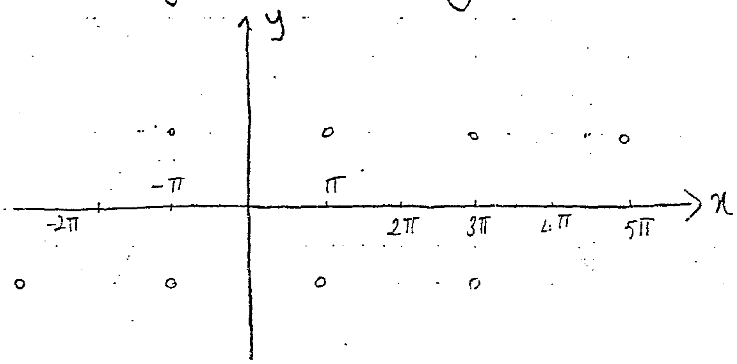
FOURIER SERİLERİ

Bir fonksiyonun trigonometrik bir seriye açılımıdır.

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+a_3\cos 3x+b_3\sin 3x+\cdots$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

Düzgün süreksiz iki nokta arasında fonksiyonun sürekli olması ve yakınsak olduğunda Fourier serisine açılabilir.



Düzgün süreksiz

$f(c)$ noktası düzgün süreksizlik noktası ise $f(c)=\dfrac{f(c-0)+f(c+0)}{2}$ 'yi vermelidir.

Teorem: 2π periyodlu bir $f(x)$ fonksiyonu sonlu sayıda düzgün süreksizlik noktası hariç $(-\pi,\pi)$ aralığında sürekli ise ve bu aralıkta ancak, sonlu sayıda extremuma sahip ise $f(x)$ fonksiyonu, x'in her değeri için yakınsak olan ve toplamı bu fonksiyona eşit bulunan bir Fourier serisi kabul eder. (Drihclet şartı)

$$\int_{-\pi}^{\pi}\cos nx\,dx=\frac{1}{n}\Big|\sin nx\Big|_{-\pi}^{\pi}=0$$

$$\int_{-\pi}^{\pi}\sin nx\,dx=\frac{-1}{n}\Big|\cos nx\Big|_{-\pi}^{\pi}=0 \qquad \cos n\pi - \cos n(-\pi)$$

$$\frac{1}{2}\int_{-\pi}^{\pi}2\cos kx.\cos nx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\left\{\cos(n+k)x+\cos(n-k)x\right\}dx$$

$$=\frac{1}{2}\left\{\Big|\frac{1}{n+k}\sin(n+k)x+\frac{1}{n-k}\sin(n-k)x\Big|_{-\pi}^{\pi}\right\}=0$$

$$\int_{-\pi}^{\pi}\sin kx.\sin nx\,dx=0$$

$$\int_{-\pi}^{\pi} \sin kx \cdot \cos nx\,dx = 0$$

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos kx \cdot \cos nx\,dx = 0 \\ &\int_{-\pi}^{\pi} \sin kx \cdot \sin nx\,dx = 0 \\ &\int_{-\pi}^{\pi} \sin kx \cdot \cos nx\,dx = 0 \end{aligned}\right\} n \neq k \qquad n = k \text{ için} \left\{\begin{aligned} &= \pi \\ &= \pi \\ &= 0 \end{aligned}\right.$$

$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2nx)\,dx$$

$$= \frac{1}{2}\left| x + \frac{1}{2n}\sin 2nx \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}(\pi+\pi)$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi$$

$$= \pi$$

Fourier serisinde a_0, a_n ve b_n'in bulunması:

$$* \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \Big\{ \underbrace{\int_{-\pi}^{\pi} a_n \cos nx\,dx}_{0} + \underbrace{\int_{-\pi}^{\pi} b_n \sin nx\,dx}_{0} \Big\}$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}(\pi+\pi)$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$* f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi} \cos kx\,dx}_{} + \sum_{n=1}^{\infty} \Big\{ \int_{-\pi}^{\pi} a_n \cos nx \cos kx\,dx + \underbrace{\int_{-\pi}^{\pi} b_n \sin nx \cos kx\,dx}_{0} \Big\}$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0 + a_k \pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

$$* \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

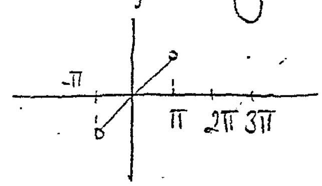
$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi}\sin kx\,dx}_{0} + \sum_{n=1}^{\infty}\Big\{ \underbrace{\int_{-\pi}^{\pi} a_n \cos nx \sin kx\,dx}_{0} + \int_{-\pi}^{\pi} b_n \sin nx . \sin kx$$

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = b_k . \pi$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$$

Örnek: $f(x)=x$ fonksiyonunu $(-\pi,\pi)$ aralığında fourier serisine açınız.

$-\pi < x < \pi$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2}\left| x^2 \right|_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x . \cos nx\,dx = \frac{1}{\pi}\left| \underbrace{(x)\left(\frac{1}{n}\sin nx\right)}_{0} - (1)\left(-\frac{1}{n^2}\cos nx\right)\right|_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{1}{\pi}\left| (x)\left(\frac{-1}{n}\cos nx\right) - \underbrace{(1)\left(-\frac{1}{n^2}\sin nx\right)}_{0}\right|_{-\pi}^{\pi} =$$

$$= \frac{-1}{n\pi}\left(\pi . \cos n\pi + \pi \cos n\pi\right)$$

$$= -\frac{2}{n}\cos n\pi$$

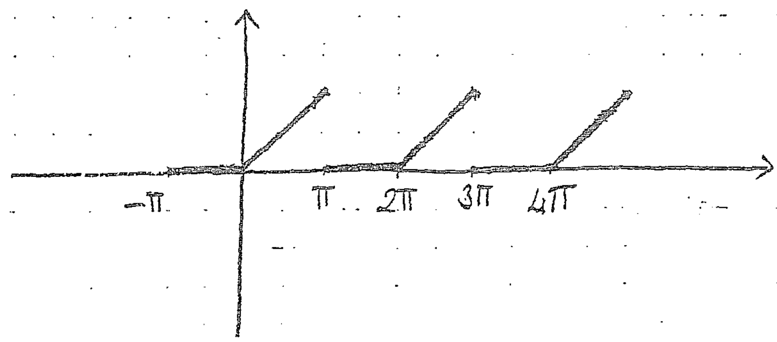
$$f(x) = x = \sum_{n=1}^{\infty} \frac{-2}{n}\cos n\pi . \sin nx$$

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \cdots\right)$$

$$\pi = 4\Big(\qquad\qquad \Big)$$

Örnek: $\left.\begin{array}{ll} -\pi < x < 0 & f(x)=0 \\ 0 < x < \pi & f(x)=x \end{array}\right\}$ şeklinde verilen fonksiyonun Fourier serisine açınız.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right\} = \frac{1}{\cancel{\pi}} \cdot \frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left\{\underbrace{\int_{-\pi}^{0} 0 \cdot \cos nx\,dx}_{0} + \int_{0}^{\pi} x \cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left|(x).\underbrace{\left(\frac{1}{n}\sin nx\right)}_{0} - (1)\left(-\frac{1}{n^2}\cos nx\right)\right|_0^{\pi}\right\}$$

$$a_n = \frac{1}{n^2\pi}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\left\{\underbrace{\int_{-\pi}^{0} 0 \cdot \sin nx\,dx}_{0} + \int_{0}^{\pi} x \cdot \sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left|(x).\left(\frac{-1}{n}\cos nx\right) - (1)\left(-\frac{1}{n^2}\sin nx\right)\right|_0^{\pi}\right\}$$

$$b_n = -\frac{1}{n\cancel{\pi}}\left[\cancel{\pi}.\cos n\pi - 0\right] = \frac{(-1)^{n+1}}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\left((-1)^n - 1\right)\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$\frac{\pi}{2} = f(x) = \frac{\pi}{4} + \left\{\left(\frac{-2}{\pi}\cos x - \frac{2}{3^2\pi}\cos 3x - \frac{2}{5^2\pi}\cos 5x - \dots\right) + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \right.\right.$$

$x = \frac{\pi}{2}$ için $f(x) = \frac{\pi}{4}$

Örnek: $-\pi < x < \pi$ $\quad f(x)=x^2$ (Çift fonk.)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx=\frac{1}{\pi}\cdot\frac{1}{3}\,x^3\Big|_{-\pi}^{\pi}=\frac{1}{\pi}\cdot\frac{1}{3}\left(\pi^3+\pi^3\right)=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left\{(x^2)\left(\frac{1}{n}\sin nx\right)-(2x)\left(-\frac{1}{n^2}\cos nx\right)+\underbrace{(2)\left(-\frac{1}{n^3}\sin nx\right)}_{0}\Big|_{-\pi}^{\pi}\right\}$$

$$a_n=\frac{1}{\pi}\left\{\frac{2}{n^2}\left(\pi\cdot\cos n\pi-(-\pi)\cos n(-\pi)\right)\right\}$$

$$a_n=\frac{4}{n^2}(-1)^n$$

$$b_n=\frac{1}{\pi}\left\{\int_{-\pi}^{\pi} x^2 \sin nx\,dx\right\}$$

$$b_n=\frac{1}{\pi}\left\{(x^2)\left(-\frac{1}{n}\cos nx\right)-\underbrace{(2x)\left(-\frac{1}{n^2}\sin nx\right)}_{0}+(2)\left(+\frac{1}{n^3}\cos nx\right)\Big|_{-\pi}^{\pi}\right\}$$

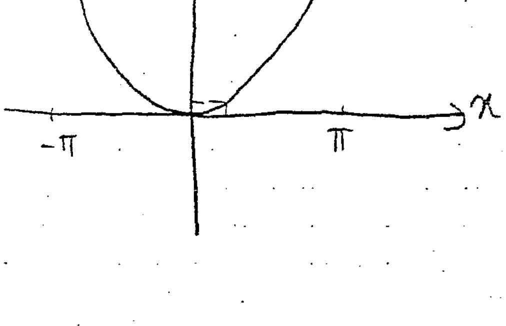
$$b_n=\frac{1}{\pi}\left\{\left[\frac{-1}{n}\underbrace{\left(\pi^2\cos n\pi-\pi^2\cos n\pi\right)}_{0}+\frac{2}{n^3}\underbrace{\left(\cos n\pi-\cos n\pi\right)}_{0}\right]\right\}=0$$

Verilen fonksiyon çift fonksiyondur denir. Çiftler hesaplanır. (a_0 ve a_n hesaplanır; $b_n=0$'dır.)

Tek fonksiyonlarda $a_0=0$, $a_n=0$'dır. b_n hesaplanır.

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\cos nx\right)$$

$$f(x)=\frac{\pi^2}{3}+4\left(-\frac{1}{1^2}\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\frac{1}{4^2}\cos 4x-\frac{1}{5^2}\cos 5x+\dots\right.$$

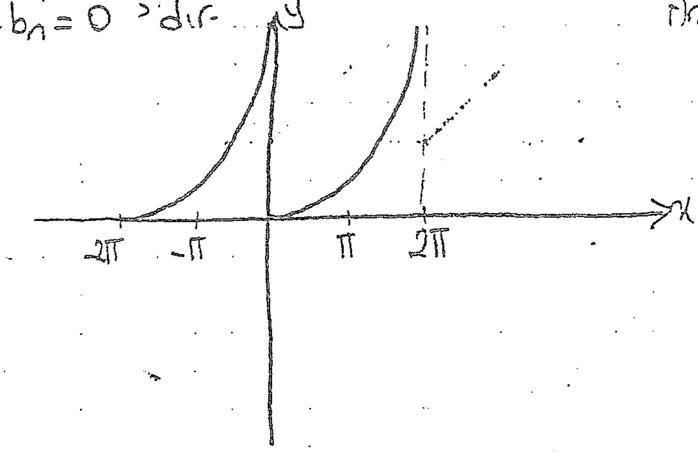


İstediğin noktada grafiklerin

* $f(x)=f(-x)$ y eksenine göre simetrik ise fonksiyon çi[ft] fonksiyondur. Aralıklar da 0 noktasına göre simetrik ise $(-\pi<x<$

$a_0=$ } hesaplanır.
$a_n=$
$b_n=0$ dır.

Aralık $(0,2\pi)$ olsaydı;



y eksenine göre simetrik olmazdı. Fonksiyon, çift fonk. olmazdı.

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Fonksiyon çiftse yarım değerde integre edebiliriz.

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n=0$$

* Hem aralık, hem nokta orijine göre simetrikse fonksiyon tek fonksiyondur.
kutba

$f(x)=-f(-x)$

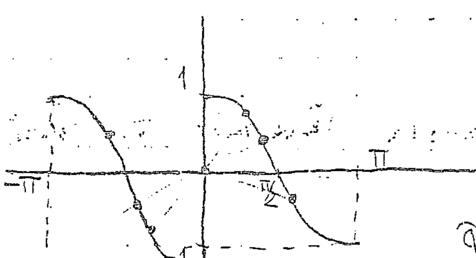
$a_0=0$

$a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cos n çift fonksiyon olmasına rağmen şekli tek fonk. dur.

<u>Örnek:</u> $f(x)=-\cos x \qquad -\pi\le x\le 0$
$f(x)=\cos x \qquad 0\le x\le \pi$



$2\cos A\cos B=\cos(A-B)+\cos(A+B)$
$2\sin A\sin B=\cos(A-B)-\cos(A+B)$
$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

Fonksiyonun tek ya da çift olduğuna karar veremiyorsak bütün katsayıları hesaplarız.

$$a_0=\frac{1}{\pi}\left\{\int_{-\pi}^{0}-\cos x\,dx+\int_0^{\pi}\cos x\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\underbrace{-\Big|\sin x\Big|_{-\pi}^{0}}_{0}+\underbrace{\Big|\sin x\Big|_{0}^{\pi}}_{0}\right\}=0$$

$$a_n=\frac{1}{\pi}\left\{-\int_{-\pi}^{0}\cos x.\cos nx\,dx+\int_{0}^{\pi}\cos x.\cos nx\,dx\right\}$$

$$a_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left[\int_{-\pi}^{0}\cos(1-n)x\,dx+\int_{-\pi}^{0}\cos(1+n)x.dx\right]+\frac{1}{2}\left[\int_{0}^{\pi}\cos(1-n)x\,dx+\int_{0}^{\pi}\cos(1+n)x\,dx\right]\right\}$$

$$a_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left[\frac{1}{1-n}\sin(1-n)x+\frac{1}{1+n}\sin(1+n)x\Big|_{-\pi}^{0}\right]+\frac{1}{2}\left[\frac{1}{1-n}\sin(1-n)x+\frac{1}{1+n}\sin(1+n)x\Big|_{0}^{\pi}\right]\right\}$$

$$b_n=\frac{1}{\pi}\left\{-\int_{-\pi}^{0}\cos x\sin nx\,dx+\int_{0}^{\pi}\cos x.\sin nx\,dx\right\}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{2}\int_{-\pi}^{0}\left[\sin(1+n)x+\sin(n-1)x\right]dx+\frac{1}{2}\int_{0}^{\pi}\left[\sin(1+n)x+\sin(n-1)x\right]dx\right\}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left|-\frac{1}{1+n}\cos(1+n)x-\frac{1}{n-1}\cos(n-1)x\right|_{-\pi}^{0}+\frac{1}{2}\left[-\frac{1}{1+n}\cos(1+n)x-\frac{1}{n-1}\cos(n-1)x\right.\right.$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left[\left(-\frac{1}{n+1}-\frac{1}{n-1}\right)-\left(-\frac{1}{1+n}(-1)^{1+n}-\frac{1}{n-1}(-1)^{n+1}\right)\right]+\frac{1}{2}\left[\left(\frac{-1}{1+n}(-1)^{1+n}-\frac{1}{n-1}(-1)^{1+n}\right)-\left(\frac{1}{1+n}-\frac{1}{n-1}\right.\right.\right.$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left[\frac{-n+1-n-1}{n^2-1}+(-1)^{1+n}\frac{n-1+1+n}{n^2-1}\right]+\frac{1}{2}\left[(-1)^{n+2}\left(\frac{n-1+1+n}{n^2-1}\right)+\left(\frac{n-1+1+n}{n^2-1}\right)\right]\right.$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{2}\left[\frac{-2n}{n^2-1}+(-1)^{1+n}\frac{2n}{n^2-1}\right]+\frac{1}{2}\left[(-1)^{n+2}\frac{2n}{n^2-1}+\frac{2n}{n^2-1}\right]\right\}$$

$$b_n=\frac{1}{\pi}\left\{\frac{n}{n^2-1}+(-1)^{1+n}\frac{n}{n^2-1}+(-1)^{n+2}\frac{n}{n^2-1}+\frac{n}{n^2-1}\right\}$$

$$b_n=\frac{1}{\pi}\,\frac{2n}{n^2-1}\qquad\qquad b_n=\frac{1}{\pi}\,\frac{8n}{(4n^2-1)}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{\pi}\frac{2n}{n^2-1}\sin nx$$

$$f(x)=\frac{2}{\pi}$$

Herhangi Periyotlu Bir Fonksiyonun Fourier Serisine Açılımı

$$x=\frac{\ell}{\pi}t\qquad(\ell\neq\pi)\qquad P=2\ell$$

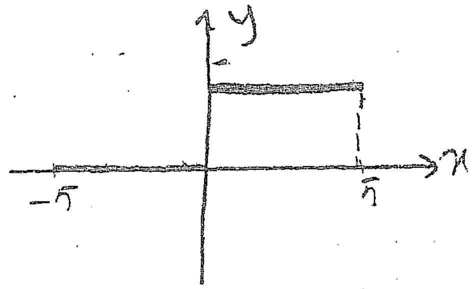
$$f(x)=f\left(\frac{\ell}{\pi}t\right)$$

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx\qquad\qquad b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x).\sin\frac{n\pi}{\ell}x\,dx$$

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x).\cos\frac{n\pi}{\ell}x\,dx$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{\ell}x+b_n \sin\frac{n\pi}{\ell}x\right)$$

<u>Örnek:</u> $-5<x<0$ $f(x)=0$; $0<x<5$ $f(x)=3$ } 10 periyotlu $f(x)$ fonksiyonunu Fourier serisine açın



$$a_0=\frac{1}{5}\left\{\int_{-5}^{0}0\,dx+\int_0^5 3\,dx\right\}=3|x|_0^5\cdot\frac{1}{5}=3$$

$$a_n=\frac{1}{5}\left\{\int_{-5}^{0}0.\cos\frac{n\pi}{5}x\,dx+\int_0^5 3.\cos\frac{n\pi}{5}x\,dx\right\}=\frac{1}{\cancel{5}}.3\left|\frac{\cancel{5}}{n\pi}\sin\frac{n\pi}{5}x\right|_0^5$$

$$a_n=\frac{3}{n\pi}(\sin n\pi-\sin 0)=0$$

$$b_n=\frac{1}{5}\left\{\int_{-5}^{0}0.\sin\frac{n\pi}{5}x\,dx+\int_0^5 3.\sin\frac{n\pi}{5}x\,dx\right\}$$

$$=\frac{1}{\cancel{5}}.3\left|\frac{-\cancel{5}}{n\pi}\cos\frac{n\pi}{5}\right|_0^5$$

$$=-\frac{3}{n\pi}(\cos n\pi-1)$$

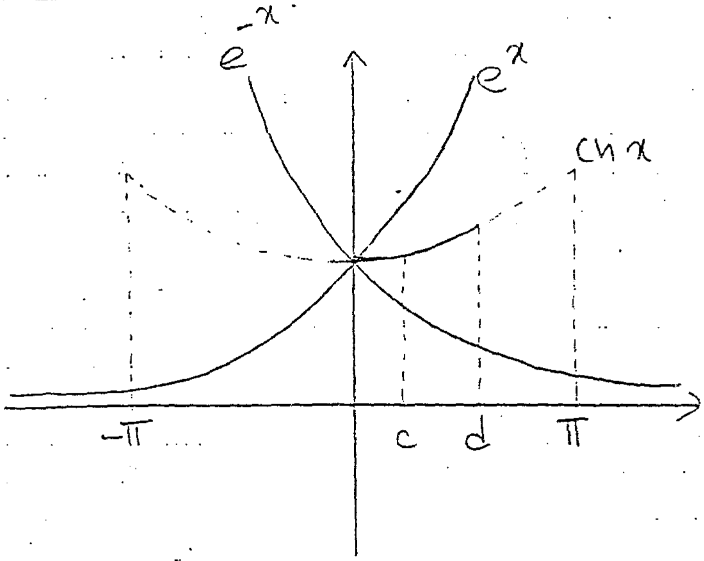
$$=-\frac{3}{n\pi}\left[(-1)^n-1\right]$$

$$f(x)=\frac{3}{2}+\sum_{n=1}^{\infty}-\frac{3}{n\pi}\left[(-1)^n-1\right]\sin\frac{n\pi}{5}x$$

$$=\frac{3}{2}-\frac{3}{\pi}\left(-\frac{2}{1}.\sin\frac{\pi}{5}x-\frac{2}{3}\sin\frac{3\pi}{5}x-\frac{2}{5}\sin\frac{5\pi}{5}x-\frac{2}{7}\sin\frac{7\pi}{5}x-\cdots\right)$$

$$=\frac{3}{2}+\frac{6}{\pi}\left(\sin\frac{\pi}{5}x+\frac{1}{3}\sin\frac{3\pi}{5}x+\frac{1}{5}\sin\frac{5\pi}{5}x+\frac{1}{7}\sin\frac{7\pi}{5}x+\cdots\right)$$

Periyodik Olmayan Bir Fonksiyonun Fourier Serisine Açılımı:



$f(x)=x^2 \qquad f(2)=4$

$f'(x)=2x \qquad f'(2)=4$

$f''(x)=2 \qquad f''(2)=2$

$x^2=4+\frac{1}{1!}(x-2)^1.4+\frac{1}{2!}(x-2)^2.2$

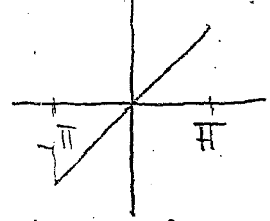
$x^2=4(x-2)^0+4(x-2)+(x-2)^2$

$4=4$

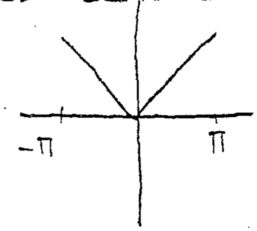
$f(x)=\frac{a_0}{2}+\Sigma(a_n \cos nx + b_n \sin nx)$

$0<x<\pi \qquad f(x)=x$

Sinüs serisine aç.

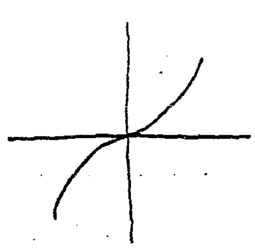


Cos serisine aç

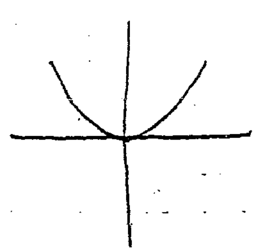


$f(x)=x^2$

Sin



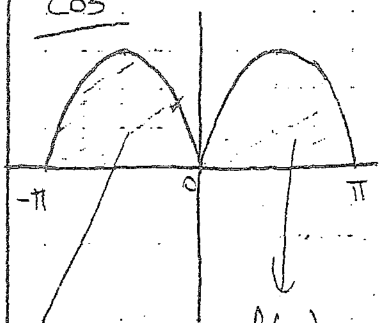
Cos



Örnek: $f(x)=x(\pi-x)$ $[0,\pi]$ aralığında Cos ve Sin serisini açınız.

Cos

Alabora top.



$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} \text{ yerine}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi} x(\pi-x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi} f^*$$

$$f(x)=x(\pi-x)=x\pi-x^2$$

$$f(x)=-x(x+\pi)$$

$-\pi<x<0$ $\quad f(x)=-x(x+\pi)$

$0<x<\pi$ $\quad f(x)=x(\pi-x)$

$$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}-x(x+\pi)\,dx+\int_0^{\pi}x(\pi-x)\,dx\right]$$

$$=\frac{1}{\pi}\left\{\left|-\frac{x^3}{3}-\frac{x^2}{2}\pi\right|_{-\pi}^{0}+\left|\frac{x^2}{2}\pi-\frac{x^3}{3}\right|_0^{\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left[0-\left(\frac{\pi^3}{3}-\frac{\pi^3}{2}\right)+\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)\right]\right\}$$

$$\boxed{a_0=\frac{\pi^2}{3}}$$

Bu tip fonksiyonlarda çift fonksiyon ise yarım aralıkta integral alınır, 2 ile çarpılır.

$$a_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(x\pi-x^2)\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{\underbrace{(x\pi-x^2)\left(\frac{1}{n}\sin nx\right)}_{0}-(\pi-2x)\left(-\frac{1}{n^2}\cos nx\right)+\underbrace{(-2)\left(-\frac{1}{n^3}\sin nx\right)}_{0}\right.$$

$$=\frac{2}{\pi n^2}\left|(\pi-2x)\cos nx\right|_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left(-\pi(-1)^n-\pi.1\right)$$

$$=\frac{2\cancel{\pi}}{\cancel{\pi}n^2}\left(-(-1)^n-1\right)$$

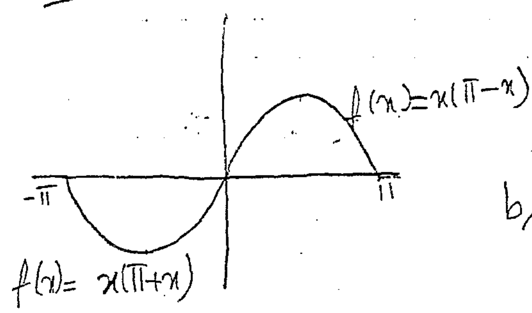
$$=\frac{2}{n^2}\left((-1)^{n+1}-1\right)$$

$$f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2}{n^2}\left[(-1)^{n+1}-1\right]\cos nx$$

$$f(x)=\frac{\pi^2}{6}+\frac{2}{2^2}(-2)\cos 2x+\frac{2}{4^2}(-2)\cos 4x+\frac{2}{6^2}(-2)\cos 6x+\cdots$$

$$f(x)=\frac{\pi^2}{6}-4\left(\frac{1}{2^2}\cos 2x+\frac{1}{4^2}\cos 4x+\frac{1}{6^2}\cos 6x+\cdots\right)$$

Sin



1. Vize: 16 Nisan

$$b_n=\frac{2}{\pi}\int_0^{\pi}(x\pi-x^2)\sin nx\,dx$$

$$b_n=\frac{2}{\pi}\Big\{\underbrace{(x\pi-x^2)\left(-\frac{1}{n}\cos nx\right)}_{0}-\underbrace{(\pi-2x)\left(-\frac{1}{n^2}\sin nx\right)}_{0}+(-2)\left(\frac{1}{n^3}\cos n\right.$$

$$b_n=-\frac{4}{\pi n^3}\Big|\cos nx\Big|_0^{\pi}$$

$$=-\frac{4}{\pi n^3}\left[(-1)^n-1\right]$$

$$f(x)=\sum_{n=1}^{\infty}-\frac{4}{\pi n^3}\left[(-1)^n-1\right]\sin nx$$

$$f(x)=\frac{8}{\pi}\left(\frac{1}{1}\sin x+\frac{1}{3^3}\sin 3x+\frac{1}{5^3}\sin 5x+\cdots\right)$$

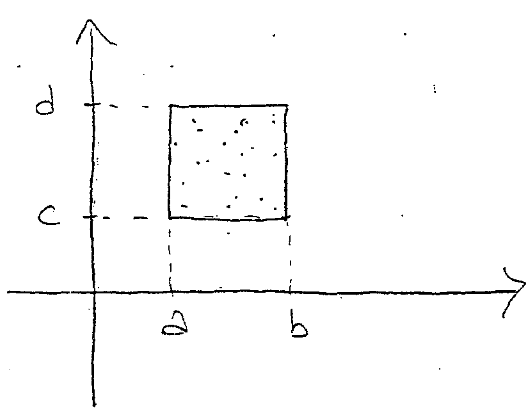
ÇOK DEĞİŞKENLİ FONKSİYONLAR

$y=f(x)$

$w=g(x,y,z,t\ldots)$

$Q=Ri^2t$

$Q=f(R,i,t)$

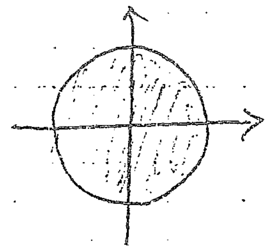


Nokta çiftlerinin oluşturduğu kapalı bölgeye o fonksiyonun domeni denir.

Örnek $f(x,y)=\log[1-(x^2+y^2)]$ fonksiyonunun domenini belirleyiniz.

$1-x^2-y^2>0$

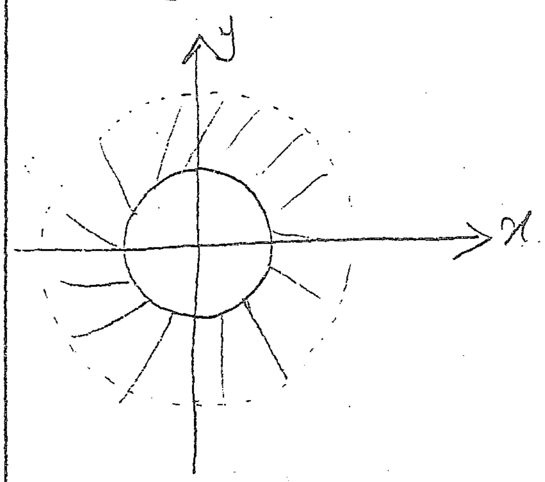
$1>x^2+y^2$



Çizgi hariç çizginin iç kısmı alınmalıdır

Örnek: $f(x,y)=\sqrt{x^2+y^2-1}+\log(4-x^2-y^2)$ alanını (tanım bölgesi) = ?

$x^2+y^2\geq 1$ $\qquad$ $4>x^2+y^2$



ÇOK DEĞİŞKENLİ FONKSİYONLARDA LİMİT VE SÜREKLİLİK

Birbirine bağlı olmaksızın $x\to a$, $y\to b$ olması demek değişken (x,y) noktasının herhangi bir yoldan gittikçe (a,b) noktasına yaklaşması demektir.

$(x,y)\to(a,b)$ veya $\lim(x,y)=(a,b)$ gösterilir ve (x,y) değişken çiftinin 2 kat limiti olarak adlandırılır. Burada gerek ve yeter şart

$$(x-a)^2+(y-b)^2\to 0$$

$$x-a\to 0$$
$$y-b\to 0$$

$z=f(x,y)$ fonksiyonunun (a,b) noktasındaki C limit değerinden şunu anlayacağız.

$\varepsilon>0$

$|x-a|<\delta \qquad |y-b|<\delta$

$|f(x,y)-c|<\varepsilon$

$\delta=\delta(\varepsilon)$ yazılabiliyorsa

$\lim\limits_{\substack{x\to a\\ y\to b}}[f(x,y)]=c$ dir denir.

Örnek: $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2xy}{x^2+y^2+2^{xy}} = \dfrac{4}{5+4} = \dfrac{4}{9}$ //

Teorem: $u=f(x,y)$, $\vartheta=g(x,y)$ fonksiyonları (xoy) düzleminin bir Δ domeninde tanımlı olsunlar.

① $\lim\limits_{\substack{x\to a\\ y\to b}} \left[f(x,y)+g(x,y)\right] = \lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) + \lim\limits_{\substack{x\to a\\ y\to b}} g(x,y) = c+d$

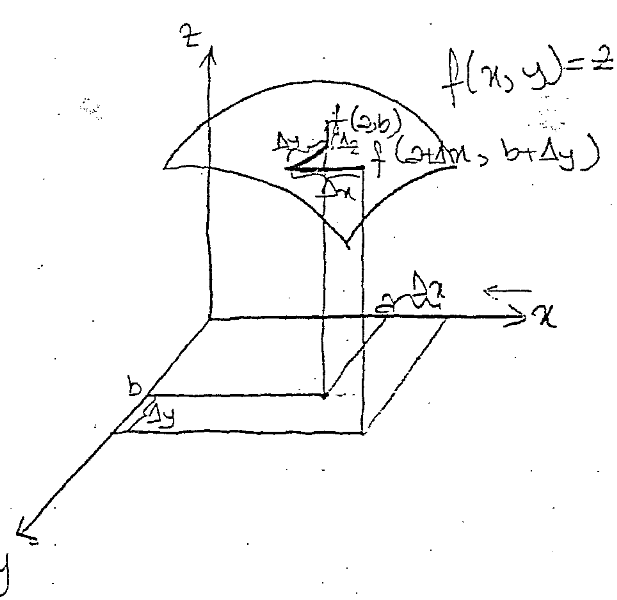
② $\lim\limits_{\substack{x\to a\\ y\to b}} \left[f(x,y)\cdot g(x,y)\right] = \lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) \cdot \lim\limits_{\substack{x\to a\\ y\to b}} g(x,y) = c\cdot d$

Çok değişkenli fonksiyonlarda da limit özellikleri geçerlidir.

Süreklilik:

$\lim\limits_{\Delta x\to 0} \Delta y = 0 \quad z=f(x,y)$

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$ ise fonksiyon verilen noktada süreklidir denir.



$a=x+\Delta x$

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \left[f(a+\Delta x, b+\Delta y) - f(a,b)\right] = 0$

Örnek: $z=x^2+y^2$ [illegible]

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \left\{\left[(x+\Delta x)^2+(y+\Delta y)^2-(x^2+y^2)\right]\right\} \stackrel{?}{=} 0$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \left(\cancel{x^2}+2x\,\Delta x+(\Delta x)^2+\cancel{y^2}+2y\Delta y+(\Delta y)^2-\cancel{x^2}-\cancel{y^2}\right)=0$$

KISMİ TÜREVLER

Eğer bir fonksiyon n değişkenli bir fonksiyonsa bu fonksiyonun $n-1$ değişkenini sabit tutarak diğer değişkene göre türevi alınırsa buna o fonksiyonun kısmi türevi denir.

$z=f(x,y)$ fonksiyonunun x'e göre kısmi türevi;

z'_x , $f'_x(x,y,z)$, $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial f}{\partial x}$

$z=x^2+5x+1$

$\dfrac{dz}{dx}=2x+5$ Dik

$z=f(x,y)$ ise [illegible]

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

Örnek: $z=x.e^{x^2y}$ fonksiyonunun x'e ve y'ye göre türevini alınız.

$$\frac{\partial z}{\partial x}=e^{x^2y}+x.e^{x^2y}.2xy$$

$$\frac{\partial z}{\partial y}=x.e^{x^2y}.x^2=x^3.e^{x^2y}$$

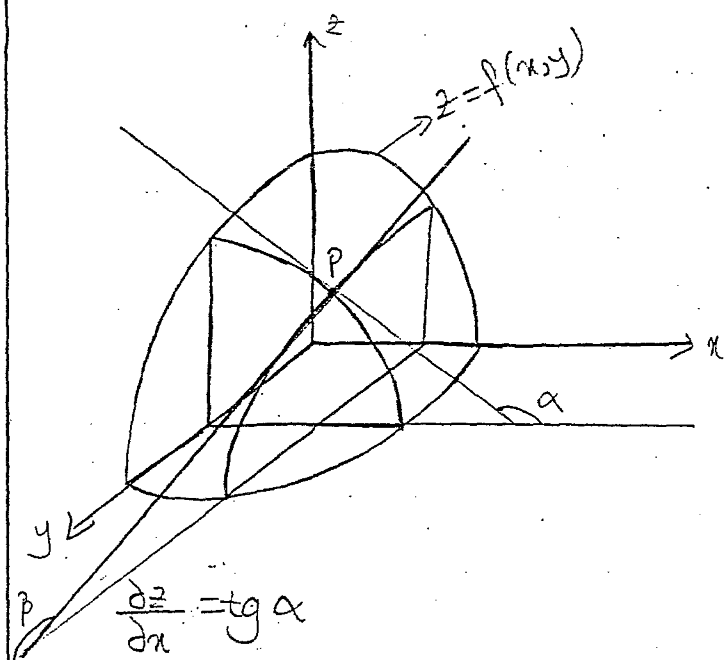
Örnek: $f(x,y,z)=y.e^{x^2}+x.\ln(y^2-z^2)+\operatorname{arctg} 3z^2$

$$\frac{\partial f}{\partial x}=y.e^{x^2}.2x+\ln(y^2-z^2)+0$$

$$\frac{\partial f}{\partial y}=e^{x^2}+x.\frac{2y}{(y^2-z^2)}+0$$

$$\frac{\partial f}{\partial z}=x.\frac{-2z}{(y^2-z^2)}+\frac{6z}{1+9z^4}$$

Kısmi Türevin Geometrik Anlamı:



$$\frac{\partial z}{\partial x} = tg\,\alpha$$

$$\frac{\partial z}{\partial y} = tg\,\beta$$

YÜKSEK MERTEBEDEN KISMİ TÜREVLER

$$\frac{\partial z}{\partial x} = z'_x$$

$$\frac{\partial z'_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z'_{xx}$$

$$\frac{\partial z'_{xx}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}$$

$$Z = f(x,y)$$

$$\frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial y^2},\ \frac{\partial^2 z}{\partial x\,\partial y},\ \frac{\partial^2 z}{\partial y\,\partial x}$$

Örnek: $Z = 3x^2y - x.\sin xy$ ifadesinde 2. mertebeye kadar olan kısmi türevlerini alınız.

$$\frac{\partial z}{\partial x} = 6xy - (\sin xy + xy.\cos xy)$$

$$\frac{\partial^2 z}{\partial x^2} = 6y - \left[y.\cos xy + \left(y.\cos xy - xy^2\sin xy\right)\right]$$

$$\frac{\partial z}{\partial y} = 3x^2 - x^2\cos xy$$

$$\frac{\partial^2 z}{\partial y^2} = x^3\sin xy$$

x'in y'ye göre türevi

$$\frac{\partial^2 z}{\partial y \partial x} = 6x - (\underbrace{x.\cos xy + x.\cos xy}_{} - x^2 y \sin xy)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x - (2x.\cos xy - x^2 y \sin xy)$$

Örnek: $u = e^x.\sin y + e^y.\sin z$ $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ olduğunu gösteriniz.

$$\frac{\partial u}{\partial x} = e^x.\sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x.\sin y$$

$$\frac{\partial u}{\partial y} = \cos y.e^x + e^y.\sin z$$

$$\frac{\partial^2 u}{\partial y^2} = -\sin y.e^x + e^y.\sin z$$

$$\frac{\partial u}{\partial z} = e^y.\cos z$$

$$\frac{\partial^2 u}{\partial z^2} = -\sin z.e^y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\cancel{e^x.\sin y} - \cancel{e^x.\sin y} + \cancel{e^y \sin z} - \cancel{e^y \sin z}$$

$$0 = 0$$

Teorem: $z = f(x,y)$ fonksiyonu ve bu fonksiyonun f_x', f_y', f''_{xy}, f_{yx} kısmi türevleri bir $M(x,y)$ noktasında tanımlı ve sürekli iseler bu nokta için

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

tir. Yani yüksek mertebeden türev alında sıra önemli değildir. (Schwarz Teoremi)

BILEŞİK FONKSİYONLARIN TÜREVLERİ

$u = f(x,y) \qquad x = x(t) \quad , y = y(t)$

$u = f[x(t), y(t)]$

$u = \varphi(t)$

$\frac{du}{dt} =$

$u = f(x,y)$

$du = f_x dx + f_y . dy$

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}$$

$$\boxed{\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}}$$

Bileşik fonksiyonun parametreye bağlı türevi.

$u=f(x,y) \quad x=x(s,\vartheta) \quad y=y(s,\vartheta)$

$$\frac{\partial u}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial \vartheta}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \vartheta}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \vartheta}$$

Örnek: $z=e^x \sin y \quad , x=\ln t, \quad y=t^2$

$\frac{dz}{dt}=?$

$z=t.\sin t^2$

$$\frac{dz}{dt}=\sin t^2+2t^2.\cos t^2$$

$$\frac{dz}{dt}=e^x.\sin y.\frac{1}{t}+e^x.\cos y.2t$$

$$y'=-\frac{f'_x}{f'_y}$$

$f(x,y)=0$

$$0=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$-f'_x dx=f'_y\, dy$$

$$\boxed{-\frac{f'_x}{f'_y}=\frac{dy}{dx}}$$

Örnek: $z=e^{xy} \quad x=\sqrt{t^2+s^2} \quad , y=\arctg\frac{s}{t}$

$z=e^{\sqrt{t^2+s^2}.\arctg\frac{s}{t}}$

$$\frac{\partial z}{\partial s}=y.e^{xy}.\frac{2s}{2\sqrt{t^2+s^2}}+x.e^{xy}.\frac{1/t}{1+\frac{s^2}{t^2}}$$

$$\frac{\partial z}{\partial t}=y.e^{xy}.\frac{2t}{2\sqrt{t^2+s^2}}+x.e^{xy}\frac{-s/t^2}{1+\frac{s^2}{t^2}}$$

Seda

TOPLAM DİFERANSİYEL

$u=f(x,y)$ fonksiyonunun x'e ve y'ye göre kısmi türevlerinin mevcut ve sürekli olduğunu kabul edelim.

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$ u fonksiyonunun toplam diferansiy

Toplam diferansiyel, her bir bağımsız değişkene göre alınmış kısmi diferansiyellerin toplamına eşittir.

$u=f(x,y,z)$ olsaydı,

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

Örnek: $z=\ln(x^2+y^2)$ dz ifadesini bulunuz

$$dz=\frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy$$

$$dz=\frac{2}{x^2+y^2}\left(x\cdot dx+y\cdot dy\right)$$

Örnek: $u=\operatorname{Arctg}\frac{xy}{z}$ $du=?$

$$du=\frac{y/z}{1+\frac{x^2y^2}{z^2}}dx+\frac{x/z}{1+\frac{x^2y^2}{z^2}}dy+\frac{\frac{-xy}{z^2}}{1+\frac{x^2y^2}{z^2}}dz$$

TAM DİFERANSİYEL OLMA ŞARTI

$(\sim\sim)dx+(\sim\sim)dy$

$$\underbrace{e^{xy}\sin y}_{M(x,y)}\cdot dx+\underbrace{\cos y}_{N(x,y)}dy$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$u=f(x,y)$

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$M(x,y)\,dx=N(x,y)dy$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \stackrel{?}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$ Tam diferansiyel olma şartı,

Örnek: $\cos y . dx + (2y - x.\sin y)dy$ ifadesinin tam bir diferansiyel olup olmadığını gösteriniz.

$$\frac{\partial}{\partial y}(\cos y) \stackrel{?}{=} \frac{\partial}{\partial x}(2y - x.\sin y)$$

$$-\sin y = -\sin y$$ Tam diferansiyeldir.

Örnek: $x^2.\sin y\, dx + x^2.\cos y\, dy$ ifadesi tam diferansiyel midir?

$$\frac{\partial}{\partial y}(x^2.\sin y) \stackrel{?}{=} \frac{\partial}{\partial x}(x^2\cos y)$$

$$x^2 \cos y \neq 2x.\cos y$$ Tam diferansiyel değil.

YÜKSEK MERTEBEDEN TOPLAM DİFERANSİYELLER

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$ x'e ve y'ye göre türevini al.

$$d(df) = d^2 f = \frac{\partial(\ )}{\partial x}dx + \frac{\partial(\ )}{\partial y}dy$$

$$= \left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 f}{\partial x \partial y}dx + \frac{\partial^2 f}{\partial y^2}dy\right)dy$$

$$= \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x \partial y}dy\,dx + \frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2$$

$$d^2 f = \left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2\right)$$

$$d^2 f = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)^2$$

$$d(d^2 f) = d^3 f = \left(\frac{\partial^3 f}{\partial x^3}dx^2 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx\,dy + \frac{\partial^3 f}{\partial x \partial y^2}dy^2\right)dx + \left(\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 + 2\frac{\partial^3 f}{\partial x \partial y^2}dx\,dy + \frac{\partial^3 f}{\partial y^3}dy^2\right)dy$$

$$= \frac{\partial^3 f}{\partial x^3}.dx^3 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + \frac{\partial^3 f}{\partial x \partial y^2}dy^2 dx + \frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + 2\frac{\partial^3 f}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$$

$$= \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dy^2 dx + \frac{\partial^3 f}{\partial y^3}dy^3$$

$$= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)^3$$

$$\boxed{d^n f = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)^n}$$ Parantez açıldıktan sonra yerine konur.

Örnek: $z = 2x^2 - 3xy - y^2 \quad d^2z = ?$

$$d^2z = \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right)^2 = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2$$

$$\frac{\partial z}{\partial x} = 4x - 3y \qquad d^2z = 4dx^2 - 6dxdy - 2dy^2$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 4}$$

$$\frac{\partial z}{\partial y} = -3x - 2y$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = -2}$$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = -3}$$

TOPLAM DİFERANSİYELİN YAKLAŞIK HESABA UYGULANMASI

Hatırlatma:

$y = f(x)$

$\Delta y = f(x + \Delta x) - f(x, y)$

$dy = f'(x) dx$

$\Delta y \cong dy$

$\Delta y = dy + \varepsilon_1 . \Delta x$

$dy \cong f(x + \Delta x) - f(x, y)$

$dy + f(x, y) = f(x + \Delta x)$

$u = f(x, y)$

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

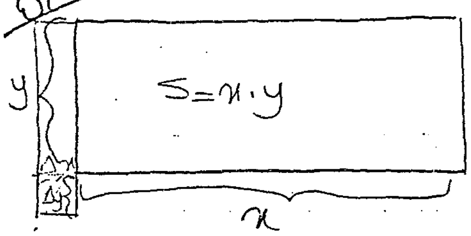
$\Delta u \cong du$

$du \cong f(x + \Delta x, y + \Delta y) - f(x, y) \Rightarrow du + f(x, y) \cong f(x + \Delta x, y + \Delta y)$

$\Delta u = du + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

minimum küçüklüklerdir.

Örnek:



Dikdörtgenin kenarlarına Δx ve Δy kadar bir artım verelim. Dikdörtgenin alanı ne kadar değişir?

$S(x+\Delta x, y+\Delta y) - S(x,y)$

$(x+\Delta x)(y+\Delta y) - x.y = \Delta S$

$\cancel{xy} + x\Delta y + y\Delta x + \Delta x \Delta y - \cancel{x.y} = \Delta S$

$x\Delta y + y\Delta x + \Delta x.\Delta y = \Delta S$ → Gerçek oynama miktarı

yaklaşık

$S = x.y$

$dS = y.dx + x.dy$

$\Delta x \Delta y$'lik kısmı hesaplamayacaktık.

Örnek: $z = 3x^2 + 2y^2$ ise $x=2$; $y=3$; $\Delta x = 0,01$; $\Delta y = 0,02$ olmak üzere dz ve Δz'i hesaplayınız.

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta z = 3(x+\Delta x)^2 + 2(y+\Delta y)^2 - (3x^2 + 2y^2)$

$\Delta z = \cancel{3x^2} + 6x\Delta x + 3(\Delta x)^2 + \cancel{2y^2} + 4y\Delta y + 2(\Delta y)^2 - \cancel{3x^2} - \cancel{2y^2}$

$\Delta z = 6x\Delta x + 3(\Delta x)^2 + 4y\Delta y + 2(\Delta y)^2$

$\Delta z = 6.2.0,01 + 3.(0,01)^2 + 4.3.0,02 + 2.(0,02)^2$

$\Delta z = 0,12 + 0,0003 + 0,24 + 0,0008$

$\Delta z = 0,3611$

$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$dz = 6x.dx + 4y\,dy$

$dz = 6.2.0,01 + 4.3.0,02$

$dz = 0,12 + 0,24$

$dz = 0,3600$

$\Delta z - dz = 0,0011$

Örnek: $\sqrt{(2,97)^2+(4,01)^2} = ?$ $(4,9909)$

$x=3$ $\quad y=4$

$\Delta x=-0,03$ $\quad \Delta y=0,01$

$z=\sqrt{x^2+y^2}$

$$dz=\frac{x}{\sqrt{x^2+y^2}}dx+\frac{y}{\sqrt{x^2+y^2}}dy$$

$$dz=\frac{3}{5}(-0,03)+\frac{4}{5}(0,01)$$

$$dz=-0,01$$

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$$

$$\Delta z+f(x,y)=f(x+\Delta x,y+\Delta y)$$

$$\Delta z+5=\sqrt{(2,97)^2+(4,01)^2}$$

$$dz+5\cong\sqrt{(2,97)^2+(4,01)^2}$$

$$-0,01+5\cong\sqrt{\quad}=4,9900$$

HATA HESABI

gerçek değer $\quad$ hesaplanan değer

$u-u_1=\Delta u \longrightarrow$ mutlak hata

$\frac{\Delta u}{u}$ oranına bağıl hata denir.

$u=f(x,y,z,---)$ fonksiyonu tanımlansın. Hata $\Delta x,\Delta y,\Delta z$ olsun. Bu mutlak hataların mutlak değerleri yeteri derecede küçükseler Δu toplam hata yerine fonksiyonun toplam diferansiyeli alınabilir.

$$\Delta u\cong\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\frac{\partial f}{\partial z}\Delta z+---$$ yaklaşık değer

elde edilir. Bu bağıntıdaki kısmi türevler ve bağımsız değişkenlere ait hatalar (+) veya (−) olabilir. Bunların yerine mutlak değeri konursa

$$|\Delta u|\leq\left|\frac{\partial f}{\partial x}\right||\Delta x|+\left|\frac{\partial f}{\partial y}\right||\Delta y|+\left|\frac{\partial f}{\partial z}\right||\Delta z|+---$$ eşitsizliği elde edilir. Bu da bize aranılan hatanın max. değerini verir.

$$\frac{|\Delta u|}{|f|}\leq\frac{\left|\frac{\partial f}{\partial x}\right|}{|f|}|\Delta x|+\frac{\left|\frac{\partial f}{\partial y}\right|}{|f|}|\Delta y|+\frac{\left|\frac{\partial f}{\partial z}\right|}{|f|}|\Delta z|+---$$

Toplam bağıl hata

$$\frac{|\Delta u|}{|u|}\leq\left|\frac{\partial}{\partial x}\ln f\right||\Delta x|+\left|\frac{\partial}{\partial y}\ln f\right||\Delta y|+\left|\frac{\partial}{\partial z}\ln f\right||\Delta z|+---$$

Örnek: $u=x+y+z$ Mutlak hatasını bulun.

$|\Delta u| \leq |\Delta x|+|\Delta y|+|\Delta z|$

Örnek: $u=x-y$ $\quad |\Delta u|=?$

$u=x+(-y)$

$|\Delta u| \leq |\Delta x|+|\Delta y|$

Örnek: $u=x.y$ $\quad |\Delta u|=?$

$|\Delta u| \leq |y||\Delta x|+|x||\Delta y|$

Örnek: $u=\dfrac{x}{y}$

$$|\Delta u| \leq \left|\frac{1}{y}\right| |\Delta x| + \left|\frac{-x}{y^2}\right| |\Delta y|$$

Örnek: Bir sarkacın l uzunluğu 0,01m hata ile $l=1m$ olarak ölçülmüştür. $\pi=3,14$ alınarak $\Delta\pi=0,005$ olarak bulunmuş, $g=9,8m/s^2$ alınarak $\Delta g=0,02\ m/s^2$ olarak bulunmuştur. Periyotta yapılan bağıl hatayı bulun. $\left(T=2\pi\sqrt{\dfrac{l}{g}}\right)$

$$\ln T=\ln 2\pi+\ln\left(\frac{l}{g}\right)^{1/2}$$

$$\ln T=\ln 2+\ln\pi+\frac{1}{2}\ln l-\frac{1}{2}\ln g$$

$$\frac{|\Delta T|}{|T|} \leq \left|\frac{\partial}{\partial l}\ln T\right| |\Delta l| + \left|\frac{\partial}{\partial \pi}\ln T\right| |\Delta\pi| + \left|\frac{\partial}{\partial g}\ln T\right| |\Delta g|$$

$$\frac{|\Delta T|}{|T|} \leq \left|\frac{1}{2l}\right| |\Delta l| + \left|\frac{1}{\pi}\right| |\Delta\pi| + \left|-\frac{1}{2}g\right| |\Delta g|$$

$$\frac{|\Delta T|}{|T|} \leq \frac{1}{2}(0,01)+\frac{1}{3,14}.(0,005)+\frac{1}{2}.9,8.(0,02)$$

$$\frac{|\Delta T|}{|T|} \leq 0,0076$$

Örnek: Dirençleri r_1, r_2, r_3 olan 3 direnç teli paralel olarak bağlanmışsa sonuç R direnci $\dfrac{1}{R}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$ bağıntısıyla verilmiştir. r_1, r_2, r_3 dirençlerinin ölçülmesinde aynı E bağıl hatası yapıldığı zaman R'nin hesabında yapılacak bağıl hatanın da E olacağını gösteriniz.

$$\frac{|\Delta r_1|}{|r_1|}=\frac{|\Delta r_2|}{|r_2|}=\frac{|\Delta r_3|}{|r_3|}=E \qquad \frac{|\Delta R|}{|R|}\stackrel{?}{=}E$$

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$

$$\left|-\frac{\Delta R}{R^2}\right|\le\left|-\frac{1}{r_1^2}\right||\Delta r_1|+\left|-\frac{1}{r_2^2}\right||\Delta r_2|+\left|-\frac{1}{r_3^2}\right||\Delta r_3|$$

$$\frac{1}{R}\left|\frac{\Delta R}{R}\right|\le\frac{1}{r_1}\left|\frac{\Delta r_1}{r_1}\right|+\frac{1}{r_2}\left|\frac{\Delta r_2}{r_2}\right|+\frac{1}{r_3}\left|\frac{\Delta r_3}{r_3}\right|$$

$$\cancel{\frac{1}{R}}\left|\frac{\Delta R}{R}\right|\le E\,\cancel{\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}$$

$$\left|\frac{\Delta R}{R}\right|\le E$$

= KAPALI FONKSİYONLARIN TÜREVİ =

$y=f(x)$

$f(x,y)=0$

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$$

$$\frac{\partial f}{\partial y}\cdot dy=-\frac{\partial f}{\partial x}dx$$

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=-\frac{f'_x}{f'_y}$$

$Z=f(x,y)$ ise Z'_x, Z'_y istenebilir. $Z=x^2+yx$ $Z'_x=2x+y$, $Z'_y=x$

$f(x,y,z)=0$ ise

f'in x'e göre tr. $\frac{\partial f}{\partial x}\cdot dx+\frac{\partial f}{\partial z}\cdot dz=0 \Longrightarrow \frac{dz}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}=-\frac{f'_x}{f'_z}$

f'in y'ye göre tr. $\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}\cdot dz=0 \Longrightarrow \frac{dz}{dy}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}=-\frac{f'_y}{f'_z}$

<u>Örnek:</u> $z^2+\frac{z}{x}-\sqrt{y^2-z^2}=0$ ise $x^2\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{1}{z}$ eşitliği geçerli midir?

$$\cancel{x^2}\cdot\left(-\frac{-z/\cancel{x^2}}{2z+\frac{z}{\sqrt{y^2-z^2}}}\right)+\frac{1}{\cancel{y}}\left(-\frac{-\frac{\cancel{y}}{\sqrt{y^2-z^2}}}{2z+\frac{z}{\sqrt{y^2-z^2}}}\right)=\frac{1}{z}$$

$$\frac{z+\frac{1}{\sqrt{y^2-z^2}}}{2z+\frac{z}{\sqrt{y^2-z^2}}}=\frac{1}{z}$$

Örnek: $\frac{z}{x} - f\left(\frac{y}{z}\right) = 0$ ise $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$ sağlanır mı

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$$x \cdot \left(\frac{-\frac{z}{x^2}}{\frac{1}{x} - f'\left(\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)} \right) + y \cdot \left(- \frac{-f'\left(\frac{y}{z}\right)\frac{1}{z}}{\frac{1}{x} - f'\left(\frac{y}{z}\right)\left(\frac{-y}{z^2}\right)} \right) \stackrel{?}{=} z$$

$$\frac{\frac{z}{x} + f'\left(\frac{y}{z}\right) \cdot \frac{y}{z}}{\frac{1}{x} + f'\left(\frac{y}{z}\right)\left(\frac{y}{z^2}\right)} \stackrel{?}{=} z$$

$$\frac{z\cancel{\left(\frac{1}{x} + f'\frac{y}{z} \cdot \frac{y}{z^2}\right)}}{\cancel{\frac{1}{x} + f'\left(\frac{y}{z}\right)\left(\frac{y}{z^2}\right)}} = z$$

DEĞİŞKEN DÖNÜŞTÜRMELERİ VE FONKSİYONEL DETERMİNANT

a) Tek Bağımsız Değişken Hali:

$y = f(x)$

$x = \varphi(t)$

$\frac{dx}{dt} = \varphi'(t)$

$$w = f\left(x, y, \frac{df}{dx}, \frac{d^2f}{dx^2}, \dots\right)$$

$$w = g\left(x(t), f[x(t)],\right.$$

$$\frac{df}{dx} = \frac{df}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\left(\frac{dx}{dt}\right)} = \boxed{\frac{dy}{dt} \cdot \frac{1}{\varphi'(t)}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{dy'}{dt} \cdot \frac{1}{\varphi'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt} \cdot \frac{1}{\varphi'(t)}\right) \cdot \frac{1}{\varphi'(t)} = \left(\frac{dy}{dt} \cdot \frac{1}{\varphi'(t)}\right)' \frac{1}{\varphi'(t)}$$

$$\frac{d^3y}{dx^3} = \frac{dy''}{dx} = \frac{dy''/dt}{dx/dt} = \frac{dy''}{dt} \cdot \frac{1}{\varphi'(t)} = \frac{d}{dt}\left[\left(\frac{dy}{dt} \cdot \frac{1}{\varphi'(t)}\right)' \frac{1}{\varphi'(t)}\right] \frac{1}{\varphi'(t)}$$

Örnek: $x = \frac{1}{t}$ dönüştürmesi halinde

$$x^2 \frac{d^2y}{dx^2} + 2x \cdot \frac{dy}{dx} + \frac{a^2}{x^2} y = 0$$

denkleminin alacağı şekli bulunuz.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt} \cdot \frac{1}{-\frac{1}{t^2}} = -\frac{dy}{dt} \cdot t^2$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{d}{dt}\left(-\frac{dy}{dt}t^2\right)\cdot\frac{1}{-\frac{1}{t^2}}$$

$$= -\left[\frac{d^2y}{dt^2}t^2 + 2t\cdot\frac{dy}{dt}\right](-t^2)$$

5r

$$= \frac{d^2y}{dt^2}t^4 + 2t^3\cdot\frac{dy}{dt}$$

$$\frac{1}{t^2}\left(y''\right) + \frac{2}{t}\left(-\frac{dy}{dt}t^2\right) + a^2t^2y = 0$$

$$\frac{1}{t^2}\left(\frac{d^2y}{dt^2}t^4 + 2t^3\cdot\frac{dy}{dt}\right) + \frac{2}{t}\left(-\frac{dy}{dt}t^2\right) + a^2t^2y = 0$$

$$\frac{d^2y}{dt^2}t^2 + \cancel{2t\frac{dy}{dt}} - \cancel{\frac{2dy}{dt}\cdot t} + a^2t^2y = 0$$

$$\boxed{\frac{d^2y}{dt^2}t^2 + a^2t^2y = 0}$$

Örnek: $x\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$

bağımlı değ. bağımsız değ.

$y = f(x)$ fonksiyonunda ek edilmiş

y bağımsız değ.

$f(y) = x$

bağımsız bağımlı

$$xy'' + y'^3 - y' = 0$$

$$y' = \frac{dy}{dx} = \boxed{\frac{1}{\left(\frac{dx}{dy}\right)}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{(dy'/dy)}{(dx/dy)} = \frac{\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)}{\left(\frac{dx}{dy}\right)} = \frac{\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}}{\left(\frac{dx}{dy}\right)} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

$$-x\cdot\left[\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right] + \frac{1}{\left(\frac{dx}{dy}\right)^3} - \frac{1}{\left(\frac{dx}{dy}\right)} = 0$$

$$-x\cdot\frac{d^2x}{dy^2} + 1 - \left(\frac{dx}{dy}\right)^2 = 0$$

$$\boxed{x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - 1 = 0}$$

b-) Birden Fazla Bağımsız Değişken Hali:

$z = f(x,y)$ $\quad x = x(u,v)$ $\quad y = y(u,v)$ $\quad u = u(x,y)$ $\quad v = v(x,y)$

$$W = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \ldots\right)$$

u, v cinsinden değeri yerine konabilir

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$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$x = u+v$
$y = u-v$
$x+y = 2u$
$\frac{x+y}{2} = u$
$\frac{1}{2} = \frac{\partial u}{\partial x}$

u veya v çekilemiyorsa:

$x = x(u,v)$ Her iki tarafın x'e göre türevi;

$$1 = \frac{\partial x}{\partial u}\boxed{\frac{\partial u}{\partial x}} + \frac{\partial x}{\partial v}\boxed{\frac{\partial v}{\partial x}}$$

$y = y(u,v)$

$$0 = \frac{\partial y}{\partial u}\boxed{\frac{\partial u}{\partial x}} + \frac{\partial y}{\partial v}\boxed{\frac{\partial v}{\partial x}}$$

$x = x(u,v)$ Her iki tarafın y'ye göre türevi;

$$0 = \frac{\partial x}{\partial u}\boxed{\frac{\partial u}{\partial y}} + \frac{\partial x}{\partial v}\boxed{\frac{\partial v}{\partial y}}$$

$y = y(u,v)$

$$1 = \frac{\partial y}{\partial u}\boxed{\frac{\partial u}{\partial y}} + \frac{\partial y}{\partial v}\boxed{\frac{\partial v}{\partial y}}$$

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 1 & \frac{\partial x}{\partial v} \\ 0 & \frac{\partial y}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}} ; \quad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} \frac{\partial x}{\partial u} & 1 \\ \frac{\partial y}{\partial u} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}} ;$$

$3x+2y=\boxed{5}$
$2x-3y=\boxed{-1}$

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 5 & 2 \\ -1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}} =$$

$$y = \frac{\Delta y}{\Delta} = \frac{\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}} =$$

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} 0 & \frac{\partial x}{\partial v} \\ 1 & \frac{\partial y}{\partial v} \end{vmatrix}}{|\Delta|} ; \quad \frac{\partial v}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial u} & 0 \\ \frac{\partial y}{\partial u} & 1 \end{vmatrix}}{|\Delta|}$$

<u>**Örnek:**</u> $x = \rho . \cos\theta$
$y = \rho . \sin\theta$ Dönüşümü yapıldığında;

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \text{? ne gelir?}$$

$V = V(x,y) \qquad x = x(\rho,\theta) \qquad y = y(\rho,\theta)$

$\rho = \rho(x,y)$ ve $\theta = \theta(x,y)$ cinsinden yazılabiliyorsa uzun yola gerek yok.

$$V=V\left(x, y, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial y^2}\right) ---$$

karelerini al topla ρ'yu bul
oranla θ'yı bul

$$\rho=\sqrt{x^2+y^2} \qquad \theta = \text{Arctg}\,\frac{y}{x}$$

$$\frac{\partial \rho}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{\rho\cos\theta}{\rho} = \cos\theta \qquad \frac{\partial \rho}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{\rho\sin\theta}{\rho}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$x=x(\rho,\theta)$ ve $y=y(\rho,\theta)$'nın x'e göre türevi

$$1 = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$0 = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$x=x(\rho,\theta)$ ve $y=y(\rho,\theta)$'nın y'ye göre türevi

$$0 = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$1 = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$1 = \cos\theta \boxed{\frac{\partial \rho}{\partial x}} - \rho\sin\theta \boxed{\frac{\partial \theta}{\partial x}} \qquad 0 = \cos\theta \boxed{\frac{\partial \rho}{\partial y}} - \rho\sin\theta \boxed{\frac{\partial \theta}{\partial y}}$$

$$0 = \sin\theta \boxed{\frac{\partial \rho}{\partial x}} + \rho\cos\theta \boxed{\frac{\partial \theta}{\partial x}} \qquad 1 = \sin\theta \boxed{\frac{\partial \rho}{\partial y}} + \rho\cos\theta \boxed{\frac{\partial \theta}{\partial y}}$$

$$\frac{\partial \rho}{\partial x} = \frac{\begin{vmatrix} 1 & -\rho\sin\theta \\ 0 & \rho\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix}}; \quad \frac{\partial \theta}{\partial x} = \frac{\begin{vmatrix} \cos\theta & 1 \\ \sin\theta & 0 \end{vmatrix}}{\rho}; \quad \frac{\partial \rho}{\partial y} = \frac{\begin{vmatrix} 0 & -\rho\sin\theta \\ 1 & \rho\cos\theta \end{vmatrix}}{\rho};$$

$$\frac{\partial \theta}{\partial y} = \frac{\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 1 \end{vmatrix}}{\rho}$$

$$\frac{\partial \rho}{\partial x} = \cos\theta; \quad \frac{\partial \theta}{\partial x} = -\frac{1}{\rho}\sin\theta; \quad \frac{\partial \rho}{\partial y} = \sin\theta; \quad \frac{\partial \theta}{\partial y} = \frac{1}{\rho}\cos\theta$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho}\cdot\cos\theta - \frac{\partial V}{\partial \theta}\cdot\frac{1}{\rho}\sin\theta$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial \rho}\sin\theta + \frac{\partial V}{\partial \theta}\frac{1}{\rho}\cos\theta$$

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$$\frac{\partial V}{\partial x}=V'_x=\frac{\partial V}{\partial \rho}\cos\theta-\frac{\partial V}{\partial \theta}\frac{1}{\rho}\sin\theta$$

$$\frac{\partial V}{\partial y}=V'_y=\frac{\partial V}{\partial \rho}\sin\theta+\frac{\partial V}{\partial \theta}\frac{1}{\rho}\cos\theta$$

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial V'_x}{\partial x}=\frac{\partial V'_x}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial V'_x}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial V'_y}{\partial y}=\frac{\partial V'_y}{\partial \rho}\frac{\partial \rho}{\partial y}+\frac{\partial V'_y}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial V'_x}{\partial x}=\left[\frac{\partial^2 V}{\partial \rho^2}\cos\theta-\left(\frac{\partial^2 V}{\partial\rho\partial\theta}\cdot\frac{1}{\rho}\sin\theta-\frac{1}{\rho^2}\sin\theta\frac{\partial V}{\partial\theta}\right)\right]\cos\theta+\left[\frac{\partial^2 V}{\partial\theta\partial\rho}\cos\theta-\frac{\partial V}{\partial\rho}\sin\theta-\left(\frac{\partial^2 V}{\partial\theta^2}\frac{1}{\rho}\sin\theta+\frac{1}{\rho}\cos\theta\frac{\partial V}{\partial\theta}\right)\right]\left(-\frac{1}{\rho}\sin\theta\right)$$

$$\frac{\partial V'_y}{\partial y}=\left[\frac{\partial^2 V}{\partial \rho^2}\sin\theta+\left(\frac{\partial^2 V}{\partial\rho\partial\theta}\frac{1}{\rho}\cos\theta-\frac{1}{\rho^2}\cos\theta\frac{\partial V}{\partial\theta}\right)\right]\sin\theta+\left[\frac{\partial^2 V}{\partial\theta\partial\rho}\sin\theta+\frac{\partial V}{\partial\rho}\cos\theta+\left(\frac{\partial^2 V}{\partial\theta^2}\cdot\frac{1}{\rho}\cos\theta-\frac{\partial V}{\partial\theta}\cdot\frac{1}{\rho}\sin\theta\right)\right]\left(\frac{1}{\rho}\cos\theta\right)$$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}$$

* $Z=Z(x,y)$ $\quad x=x(u,v)\quad y=y(u,v)\quad u=u(x,y)\quad v=v(x,y)$

$Z=\varphi(x,y)$ $\qquad \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}=?$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix}\frac{\partial x}{\partial u} & 1\\ \frac{\partial y}{\partial u} & 0\end{vmatrix}}{\Delta_1}$$

$$\frac{\partial v}{\partial x}=\frac{-\frac{\partial y}{\partial u}}{\Delta_1}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

$u=u(x,y)$ ve $v=v(x,y)$'nin u'ya göre türevleri

$$1 = \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u}$$

$$0 = \frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u}$$

v'ye göre türev

$$0 = \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v}$$

$$1 = \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v}$$

$$\Delta_1 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{D(x,y)}{D(u,v)}$$

$$\Delta_2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{D(u,v)}{D(x,y)}$$

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & \frac{\partial u}{\partial y} \\ 0 & \partial v/\partial y \end{vmatrix}}{\Delta_2} = \frac{\frac{\partial v}{\partial y}}{\Delta_2}$$

$$\frac{\partial y}{\partial u} = \frac{-\frac{\partial v}{\partial x}}{\Delta_2}$$

$$\frac{\partial x}{\partial v} = \frac{-\frac{\partial u}{\partial y}}{\Delta_2}$$

$$\frac{\partial y}{\partial v} = \frac{\frac{\partial u}{\partial x}}{\Delta_2}$$

[illegible] değişkenlere göre kısmi türevlerden elde edilmiş [illegible] jakobiyen denir.

Genel anlamda fonksiyon $Z = Z(x_1, x_2, x_3, \ldots, x_n)$ bağlı olabilir.

$x_1 = x_1(u_1, u_2, \ldots, u_n)$

$x_2 = x_2(u_1, u_2, \ldots, u_n)$

— — — — — —

$x_n = x_n(u_1, u_2, \ldots, u_n)$

$$|J| = \frac{D(x_1, x_2, \ldots, x_n)}{D(u_1, u_2, \ldots, u_n)}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Örnek: $\left.\begin{array}{l} Z = u^3 + v^3 \\ x = u + v \\ y = u^2 + v^2 \end{array}\right\} \Rightarrow \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ ifadelerini u ve v cinsinden belirtiniz.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

bilemiyoruz

$$3u^2 = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot 2u$$

$$3v^2 = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot 2v$$

$$\frac{\partial z}{\partial x} = \frac{\begin{vmatrix} 3u^2 & 2u \\ 3v^2 & 2v \end{vmatrix}}{\begin{vmatrix} 1 & 2u \\ 1 & 2v \end{vmatrix}} = \frac{6(u^2v - v^2u)}{2(v-u)} = \frac{6uv(u-v)}{-2(v-u)} = -3uv$$

$$\frac{\partial z}{\partial y} = \frac{\begin{vmatrix} 1 & 3u^2 \\ 1 & 3v^2 \end{vmatrix}}{2(v-u)} = \frac{3(v^2-u^2)}{2(v-u)} = \frac{3}{2}(v+u)$$

FONKSİYONEL DETERMİNANTIN ÖZELLİKLERİ

1) $u = f_1(x,y,z) \qquad v = f_2(x,y,z) \qquad w = f_3(x,y,z)$

$x = \varphi_1(X,Y,Z) \qquad y = \varphi_2(X,Y,Z) \qquad z = \varphi_3(X,Y,Z)$

olduğuna göre

$$\frac{D(u,v,w)}{D(X,Y,Z)} = \frac{D(u,v,w)}{D(x,y,z)} \cdot \frac{D(x,y,z)}{D(X,Y,Z)} \quad \text{dir.}$$

$$\left(\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} & \end{vmatrix} \right)$$

$$\begin{vmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial X} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial X} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial X} & - & - & - \end{vmatrix}$$

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial X} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial X}$$

2-) $u = f_1(x,y,z) \qquad v = f_2(x,y,z) \qquad w = f_3(x,y,z)$

$x = \varphi_1(X,Y,Z) \qquad y = \varphi_2(X,Y,Z) \qquad z = \varphi_3(X,Y,Z)$

$$\frac{D(x,y,z)}{D(u,v,w)} \cdot \frac{D(u,v,w)}{D(x,y,z)} = 1$$

birbirlerine bağlı olan değişkenlerin jakobiyenleri çarpımı 1dir

kısmi türevlerin det.

$$\left(\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \right)$$

3-) Herbiri n değişkenli n fonksiyonun bir bağıntı ile birbirine bağlı olabilmesi için gerek ve yeter şart; bu fonksiyonların jakobiyeninin 0 olmasıdır.

$u=f_1(x,y,z) \quad v=f_2(x,y,z) \quad w=f_3(x,y,z)$

$$\frac{D(u,v,w)}{D(x,y,z)}=0$$

<u>Örnek:</u> $u=\sqrt{xy} \quad v=e^{-xy}+xy$ $\quad$ u ile v arasında bir bağıntı kurabilir miyiz?

$$\boxed{v=e^{-u^2}+u^2}$$

$$\frac{D(u,v)}{D(x,y)}=|J|=0 \qquad \boxed{u'_x=\frac{du}{dx}} \text{ olmak üzere}$$

$$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix}=\begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ -ye^{-xy}+y & -xe^{-xy}+x \end{vmatrix}=\frac{1}{2\sqrt{xy}}\underbrace{\left(\cancel{-xye^{-xy}}+\cancel{xy}+\cancel{xye^{-xy}}-\cancel{xy}\right)}_{0}=0$$

<u>Örnek:</u>

$x=\rho.\cos\theta$
$y=\rho.\sin\theta$

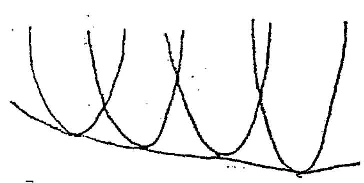
$$\frac{D(x,y)}{D(\rho,\theta)}=\begin{vmatrix} x'_\rho & x'_\theta \\ y'_\rho & y'_\theta \end{vmatrix}=\begin{vmatrix} \cos\theta & -\rho.\sin\theta \\ \sin\theta & \rho.\cos\theta \end{vmatrix}=\rho \quad \text{Bağıntı yazılamaz.}$$

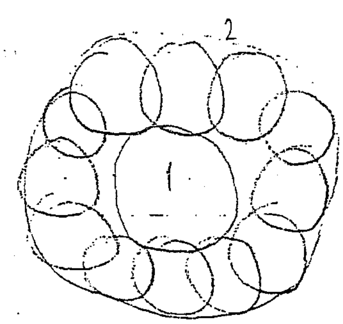
$$x^2+y^2=\rho^2$$

$$\iint dx\,dx=\iint |J|\,d\rho\,d\theta$$

DÜZLEM EĞRİLERDE ZARF

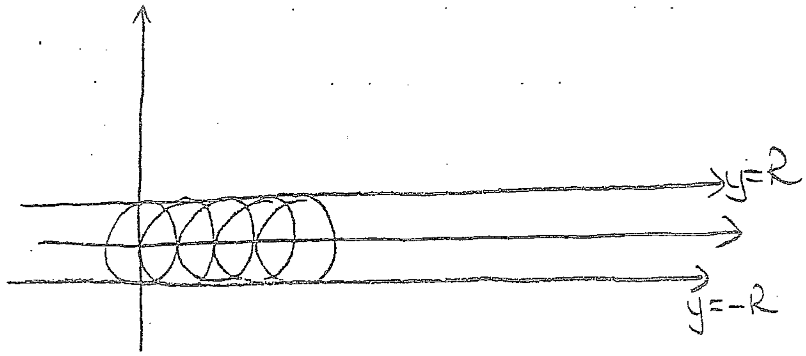
Bir eğri ailesinin herbirine aynı anda teğet olan eğriye o eğri ailesinin zarfı denir.





Zarfın parametrik denklemlerini bulmak için!
1-) Eğri ailesinin denkleminin parametreye göre türevi alınır.
2-) Türev denkleminden x ve y'ler parametreye bağlı olarak çekilerek eğri ailesinin denkleminde yerine konur.
Parametre yok edilebiliyorsa x ve y'ye bağlı kartezye denklem elde edilir.

Örnek: $(x-\alpha)^2+y^2=R^2$



$$2(x-\alpha)(-1)=0$$

$$x=\alpha$$
$$y^2=R^2$$
$$y=\mp R$$

Bu ailenin zarfı doğrudur.

Örnek: $x.\cos\alpha+y.\sin\alpha=p$

$$-x.\sin\alpha+y.\cos\alpha=0$$

$$y.\cos\alpha=x.\sin\alpha$$
$$\frac{y}{x}=\frac{\sin\alpha}{\cos\alpha}$$
$$\frac{y}{x}=tg\,\alpha$$
$$\alpha=Artg\,\frac{y}{x}$$

$$x.\cos\Big(\underbrace{Artg\,\frac{y}{x}}_{t}\Big)+y.\sin\Big(\underbrace{Artg\,\frac{y}{x}}_{t}\Big)=p$$

II. Yol

$$\sin\alpha\,/\;x.\cos\alpha+y.\sin\alpha=p$$
$$\cos\alpha\,/\;-x.\sin\alpha+y.\cos\alpha=0$$

$$y(\sin^2\alpha+\cos^2\alpha)=p.\sin\alpha$$
$$y=p.\sin\alpha$$

$$\cos\alpha\,/\;x.\cos\alpha+y.\sin\alpha=p$$
$$-\sin\alpha\,/\;-x.\sin\alpha+y.\cos\alpha=0$$

$$x(\cos^2\alpha+\sin^2\alpha)=p.\cos\alpha$$
$$x=p.\cos\alpha$$

$$\Downarrow$$

$$x=p.\cos\alpha$$
$$y=p.\sin\alpha$$
$$\boxed{x^2+y^2=p^2}$$

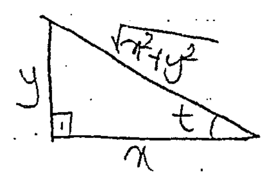
Bu eğri ailesinin zarf çemberdir.

I. Yol (Devam)

$$x.\cos t + y.\sin t = p \qquad t = \mathrm{Arctg}\,\frac{y}{x}$$

$$x.\frac{x}{\sqrt{x^2+y^2}} + y.\frac{y}{\sqrt{x^2+y^2}} = p \qquad tg\,t = \frac{y}{x}$$

$$\sqrt{x^2+y^2}\;\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = p$$



$$\sqrt{x^2+y^2} = p$$

İKİ DEĞİŞKENLİ FONKSİYONLARDA TAYLOR VE MACLOREN AÇILIMI

Taylor

$$f(x) = f(a) + \frac{1}{1!}(x-a)^1 f'(a) + \frac{1}{2!}(x-a)^2.f''(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

İki değişkenli fonksiyonlarda $\begin{matrix} x-a \\ y-b \end{matrix}$ civarında seriye açılabili

$x = a+h$

$x-a = h$

$$f(a+h) = f(a) + \frac{1}{1!}h.f'(a) + \frac{1}{2!}h^2.f''(a) + \frac{1}{3!}h^3.f'''(a) + \dots + \frac{1}{n!}h^n f^{(n)}(a) + \dots$$

$x = a+h$

$y = b+k$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)f$$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a).\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]f + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f + \dots +$$

$$\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f + \dots$$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right]f(a,b) + \frac{1}{2!}\left[h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right]^2 f(a,b) + \dots +$$

$$\frac{1}{n!}\left[h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right]^n f(a,b) + \dots$$

Örnek: $(1,-1)$ noktası civarında $f(x,y) = e^{x+y}$ 'yi Taylor serisine açınız (2. türeve kadar açınız.)

$$f(x,y) = f(a,b) + \frac{1}{1!}\left(h.\frac{\partial f}{\partial x} + k.\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2.h.k\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{3!}(\dots$$

$$e^{x+y} = 1 + (h+k) + \frac{1}{2!}(h^2 + 2hk + k^2) + \dots$$

$$e^{x+y} = f(x,y) = 1 + [(x-1) + (y+1)] + \frac{1}{2!}\left[(x-1)^2 + 2(x-1)(y+1) + (y+1)^2\right] + \dots$$

$f(1,-1)=e^0=1$

$f'_x=e^{x+y} \Rightarrow f'_x(1,-1)=e^0=1$

$f''_{xx}=e^{x+y} \Rightarrow f''_{xx}(1,-1)=e^0=1$

$f'_y=e^{x+y} \Rightarrow f'_y(1,-1)=e^0=1$

$f''_{yy}=e^{x+y} \Rightarrow f''_{yy}(1,-1)=e^0=1$

$f'_{xy}=e^{x+y} \Rightarrow f'_{xy}(1,-1)=1$

Maclorin

$$f(x,y)=f(0,0)+\frac{1}{1!}\left(x\cdot\frac{\partial}{\partial x}+y\cdot\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x\cdot\frac{\partial}{\partial x}+y\cdot\frac{\partial}{\partial y}\right)^2 f(0,0)+\dots$$

$$\frac{1}{n!}\left(x\cdot\frac{\partial}{\partial x}+y\cdot\frac{\partial}{\partial y}\right)^n f(0,0)+\dots$$

<u>Örnek:</u> $f(x,y)=e^x\cdot\sin y$ Maclorenle açınız. (3. mertebe türeve kadar)

$f(0,0)=0$

$f'_x(x,y)=e^x\cdot\sin y \Rightarrow f'_x(0,0)=0$

$f'_{xx}(x,y)=e^x\cdot\sin y \Rightarrow f'_{xx}(0,0)=0$

$f'_{xxx}(x,y)=e^x\cdot\sin y \Rightarrow f'_{xxx}(0,0)=0$

$f'_y(x,y)=e^x\cdot\cos y \Rightarrow f'_y(0,0)=1$

$f'_{yy}(x,y)=-e^x\cdot\sin y \Rightarrow f'_{yy}(0,0)=0$

$f'_{yyy}(x,y)=-e^x\cdot\cos y \Rightarrow f'_{yyy}(0,0)=-1$

$f'_{xy}(x,y)=e^x\cdot\cos y \Rightarrow f'_{xy}(0,0)=1$

$f'_{xxy}(x,y)=e^x\cdot\cos y=1$

$f'_{yyx}(x,y)=e^x\cdot\sin y=0$

$$e^x\cdot\sin y=y+\frac{1}{2}(2xy)+\frac{1}{6}(3x^2y-y^3)+\dots$$
$$=y+xy+\frac{1}{2}x^2y-\frac{1}{6}y^3+\dots$$

İKİ DEĞİŞKENLİ FONKSİYONLARDA MAXIMUM – MINIMUM

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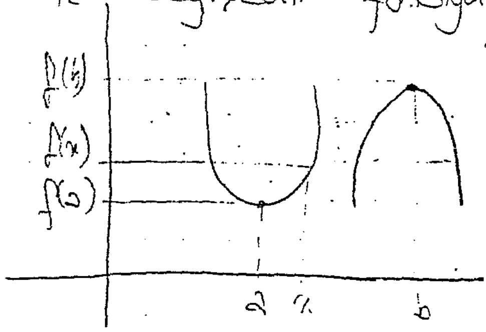
Tek değişkenli fonksiyonlarda;



$f(x) > f(a)$

$f(x) - f(a) > 0$ Min.

$f(x) < f(b)$

$f(x) - f(b) < 0$ Max.

Çift değişkenli fonksiyonlarda;

$f(x,y) - f(a,b) > 0$ Min.

$f(x,y) - f(a,b) < 0$ Max.

$Z = x^2 + y^2$

$xz + z^2 + xyz = 0$

$f(x,y,z) = 0$

$y = f(x)$

$f(x,y) = z$

$f(x,y)$ fonksiyonunu Taylor serisine açalım.

$$f(x,y) = f(a,b) + \left(h \cdot \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right) + \dots$$

İki değişkenli fonksiyonlarda max ve min bulunurken

1-) Verilen $f(x,y)$ fonksiyonunun x'e ve y'ye göre türevleri alınarak 0'a eşitlenir.

$$f'_x(x,y) = 0 \qquad f'_y(x,y) = 0 \qquad \begin{cases} A_1(x,y) \\ A_2(x,y) \\ A_3(x,y) \\ \vdots \\ A_n(x,y) \end{cases}$$

Bulduğumuz noktalar bir ekstrem nokta olmaya namzet noktalardır.

2-) $$\Delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \quad \begin{cases} \Delta < 0 \text{ ise max veya min vardır.} \\ \Delta > 0 \text{ ise max veya min aranmaz.} \\ \Delta = 0 \text{ ise şüpheli hal.} \end{cases}$$

3) 3. adıma ancak Δ'nın negatif olması halinde bakılır.

$\Delta < 0$ ise

$\frac{\partial^2 f}{\partial x^2}$ veya $\frac{\partial^2 f}{\partial y^2}$ işaretine bakılır. Biri negatif ise nokta max, pozitif ise nokta minimumdur denir.

$$\underbrace{f(x,y)-f(a,b)}=\left(h\cdot\underbrace{\frac{\partial f}{\partial x}}_{0}+k\cdot\underbrace{\frac{\partial f}{\partial y}}_{0}\right)+\frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)$$

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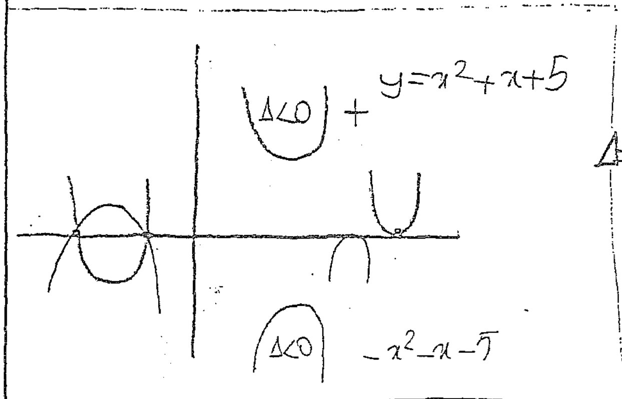
$$f(x,y)-f(a,b)=\frac{1}{2}k^2\left(\frac{h^2}{k^2}\overset{A}{\frac{\partial^2 f}{\partial x^2}}+\frac{2h}{k}\overset{B}{\frac{\partial^2 f}{\partial x\partial y}}+\overset{C}{\frac{\partial^2 f}{\partial y^2}}\right)\qquad \frac{h}{k}=t$$

$$f(x,y)-f(a,b)=\frac{1}{2}k^2\left[t^2A+2tB+C\right]$$ Bu ifadenin Δ'sına bakılı



$B^2-AC<0$ ise max, min var

$$\Delta\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}<0$$

Örnek: $f(x,y)=8x^3+y^3-12xy+8$ fonksiyonunun varsa min-max noktalarını bulunuz.

$\frac{\partial f}{\partial x}=f'_x(x,y)=24x^2-12y=0$

$\frac{\partial f}{\partial y}=f'_y(x,y)=3y^2-12x=0$

$y=\sqrt{4x}$

$24x^2-12\sqrt{4x}=0$

$x^2-\sqrt{x}=0 \qquad x^4-x=0$

$x=0 \quad x=1 \qquad x(x^3-1)=0$

$y=0 \quad y=2$

$A(0,0) \quad B(1,2)$

$\frac{\partial^2 f}{\partial x\partial y}=-12$

$\frac{\partial^2 f}{\partial x^2}=48x$

$\frac{\partial^2 f}{\partial y^2}=6y$

$\Delta=144-288xy$

A noktası için; $\Delta=144>0$ max, min yok.

B noktası için; $\Delta=144-288.1.2=144-576<0 \Rightarrow \Delta<0$

$$\frac{\partial^2 f}{\partial y^2} = 6y$$

B noktası için $\frac{\partial^2 f}{\partial y^2} = 12 > 0$ $B(1,2)$ bir minimum noktadır.

Örnek: $f(x,y) = z = (x^2+y^2) \cdot e^{-(x^2+y^2)}$ fonksiyonunun max ve min değerlerini bulunuz.

$$\frac{\partial z}{\partial x} = z'_x(x,y) = 2x \cdot e^{-(x^2+y^2)} + (-2x) \cdot e^{-(x^2+y^2)} \cdot (x^2+y^2) = 0$$

$$\frac{\partial z}{\partial y} = z'_y(x,y) = 2y \cdot e^{-(x^2+y^2)} + (-2y) \cdot e^{-(x^2+y^2)} \cdot (x^2+y^2) = 0$$

$$2x \cdot e^{-(x^2+y^2)} \quad (1-x^2-y^2) = 0$$
$$2y \cdot e^{-(x^2+y^2)} \quad (1-x^2-y^2) = 0$$

$x^2+y^2=1$ $\quad x=0$ için $y=\mp 1$ $\quad A_1(0,1)$, $A_2(0,-1)$

$\quad y=0$ için $x=\mp 1$ $\quad A_3(1,0)$, $A_4(-1,0)$

$$\left.\begin{array}{l} z_{xx} \\ z_{yy} \\ z_{xy} \end{array}\right\} \quad (z_{xy})^2 - z_{xx} \cdot z_{yy} < 0$$

Örnek: Toplamları bir a pozitif sayısına eşit çarpımları max olan üç pozitif sayıyı bulunuz.

$x+y+z=a$

$(x \cdot y \cdot z) \Rightarrow Max$

$f(x,y) = x \cdot y\,(a-x-y)$

$f(x,y) = axy - x^2y - xy^2$

$f'_x = ay - 2xy - y^2 = 0$ $\qquad y(a-2x-y)=0$

$f'_y = ax - x^2 - 2xy = 0$ $\qquad x(a-x-2y)=0$

$x=0$, $y=0$

$A_1(0,0)$ $\quad A_3(0,a)$ $\quad A_5(a,0)$

$A_2\left(\frac{a}{3}, \frac{a}{3}\right)$ $\quad A_4\left(0, \frac{a}{2}\right)$ $\quad A_6\left(\frac{a}{2}, 0\right)$

$$a - 2x - y = 0$$
$$2/\ a - x - 2y = 0$$
$$-a + 3y = 0$$
$$\boxed{y = \frac{a}{3}}$$
$$\boxed{x = \frac{a}{3}}$$

$$a - 2x - y = 0$$
$$a - x - 2y = 0$$
$$2a - 3x - 3y = 0$$

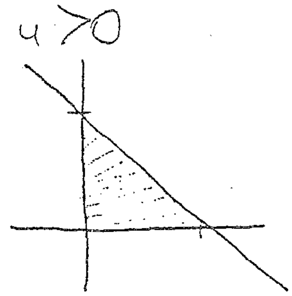
$$u = f(x,y) = x \cdot y\,(a - x - y)$$

$x > 0$

$y > 0$

$x + y < a$

$u > 0$

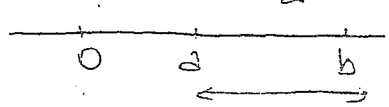


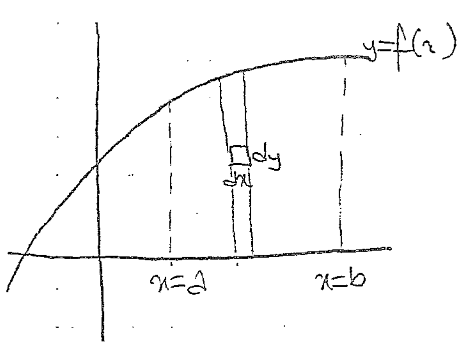
Her nokta için $(z''_{xy})^2 - z''_{xx} \cdot z''_{yy} < 0$ Noktaları tek tek yerine koy

Cevap: $\left(\frac{a}{3}, \frac{a}{3}\right)$'te bir max. kabul ediyor.

KATLI İNTEGRALLER

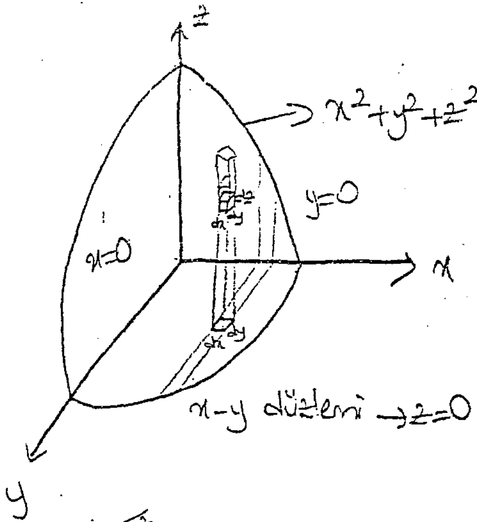
$\int_a^b dx$ → uzunluk birimi





$$\int_a^b \int_0^{f(x)} dx\,dy = \int_a^b y \Big|_0^{f(x)} dx = \int_a^b f(x)\,dx \quad \text{(Minimum cetvelin alanı)}$$

Bölge içinde seçilecek en küçük alanın bölge içinde gezdirilmesi.



$x^2+y^2+z^2=r^2$ (Merkezcil bir kürenin analitik ifadesi)

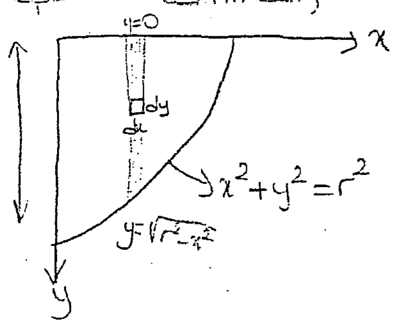
$z=\sqrt{r^2-x^2-y^2}$

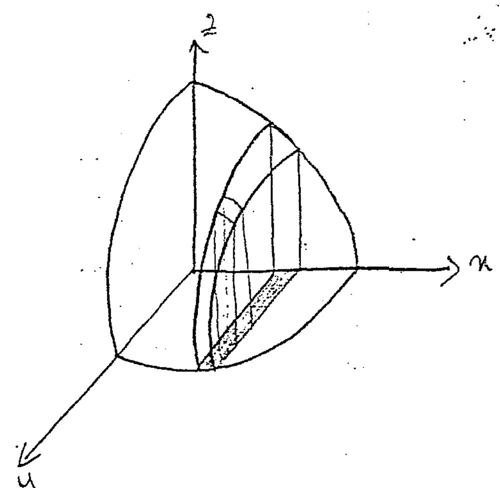
dz, küre yüzeyine kadar gider.

$$\int_{x=0}^{x=r}\int_{y=0}^{y=\sqrt{r^2-x^2}}\int_{0=z}^{\sqrt{r^2-x^2-y^2}=z} dx\,dy\,dz$$

Hangi sınır değerlerinde daha fazla değişken varsa integrale oradan başlanır.

Tepeden bakılırsa;





$$\int_{x=0}^{x=r}\int_{y=0}^{y=\sqrt{r^2-x^2}}\int_{z=0}^{z=\sqrt{r^2-x^2-y^2}} dx\,dy\,dz = \iint \left. z \right|_0^{\sqrt{r^2-x^2-y^2}} dx\,dy$$

$$\int_0^r\int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\;dx\,dy$$

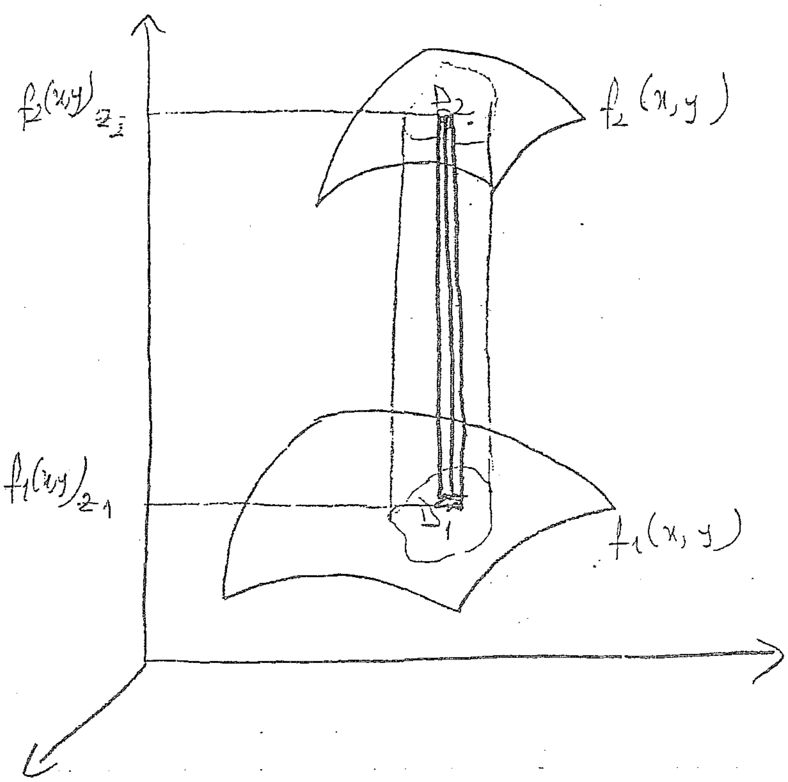
$$\int_a^b \int_c^{g(x)} f(x,y)\,dx\,dy = H$$

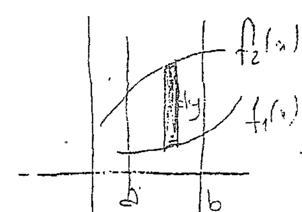
S/

$\int_a^b dx \rightarrow$ uzunluk $\quad \int_a^b \int_c^d dx\,dy \rightarrow$ Alan $\quad \int_a^b f(x)\,dx \rightarrow$ Alan

$\iiint dx\,dy\,dz \rightarrow$ Hacim

$$\underbrace{\iint f(x,y)\,dx\,dy}_{\text{Çubuğun hacmi}}$$





$$\int_a^b \int_{f_1(x)}^{f_2(x)} dx\,dy = \int [f_2(x) - f_1(x)]$$

$\iiint dx\,dy\,dz$

$\iint f(x,y)\,dy\,dx$

$$\int_a^b \int_c^{g(x)} \left[f_2(x,y) - f_1(x,y)\right] dx\,dy$$

Eğer integral sınırlarını şekle ~~bakarak~~ biz yazıyorsak integral alırken sıra önemli değildir.
Sınır değerleri veriliyorsa integral içten dışa doğrudur.

$\iint dxdy \rightarrow$ yer önemli değil

$$\int_0^{x}\int_{x}^{\sqrt{x}}\int_{1-x}^{xy} dx\,dy\,dz$$
$\downarrow$ 1 $\quad\downarrow$ 2 $\quad\downarrow$ 3

Örnek Kürenin hacmi:

$$8.\int_0^{r}\int_0^{\sqrt{r^2-x^2}}\int_0^{\sqrt{r^2-x^2-y^2}} dz\,dy\,dx = 8.\int_0^{r}\int_0^{\sqrt{r^2-x^2}} z\Big|_0^{\sqrt{r^2-x^2-y^2}} dy\,dx$$

$$=8\int_0^{r}\int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\;dy\,dx$$

$$=8\int_0^{r}\left(\int_0^{A} \sqrt{A^2-y^2}\;dy\right)dx$$

$y=a.\sin\theta$
$dy=a\cos\theta\, d\theta$

$$=8\int_0^{r}\left(\int_0^{\pi/2}\sqrt{A^2-A^2\sin^2\theta}\;A\cos\theta\, d\theta\right)dx$$

$$=8\int_0^{r}\left(A^2\int_0^{\pi/2}\cos^2\theta\, d\theta\right)dx\,.$$

Walis Teoremi

$$\int_0^{\pi/2}\cos^6 x\,dx=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}$$

$$\int_0^{\pi/2}\sin^5 x\,dx=\frac{2}{3}\cdot\frac{4}{5}$$

$$\int_0^{\pi/2}\sin^2 x\,dx=\frac{\pi}{2}\cdot\frac{1}{2}$$

$$=8.\frac{\pi}{4}\int_0^{r}A^2dx$$

$$=2\pi\int_0^{r}(r^2-x^2)\,dx$$

$$=2\pi\left|r^2x-\frac{x^3}{3}\right|_0^{r}$$

$$=2\pi\left(r^3-\frac{r^3}{3}\right)$$

$$=\frac{4r^3\pi}{3}$$

Örnek: $\displaystyle\int_0^1\int_0^{1-x} x^2y\,dy\,dx = ?$

$$\int_0^1\int_0^{1-x} x^2y\,dy\,dx = \int_0^1 \left|\frac{1}{2}x^2y^2\right|_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[x^2(1-x)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 (x^2-2x^3+x^4)\,dx$$

$$= \frac{1}{2}\left|\frac{x^3}{3}-\frac{2x^4}{4}+\frac{x^5}{5}\right|_0^1$$

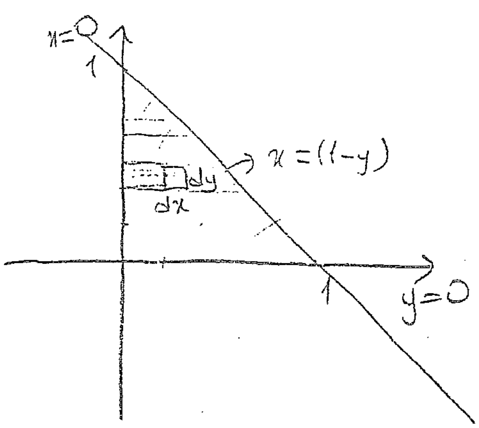
$$= \frac{1}{2}\left(\frac{1}{3}-\frac{\cancel{2}^{1}}{\cancel{4}_{2}}+\frac{1}{5}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{30}$$

$$= \frac{1}{60}$$

Örnek: D: $x=0$, $y=0$, $x+y=1$ bölgesinde $z=xy$'yi integre edin.

$$\int_0^1\int_0^{1-y} xy\,dx\,dy = \frac{1}{2}\int_0^1 yx^2\Big|_0^{1-y} dy = \frac{1}{2}\int_0^1 y(1-y)^2\,dy = \frac{1}{24}$$



Örnek: $\int_0^1 \left(\int_x^{\sqrt{x}} f(x,y)\,dy \right) dx$ integral alma sırasını değiştiriniz.

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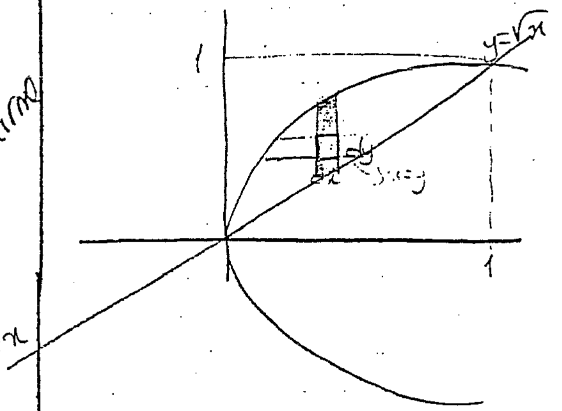
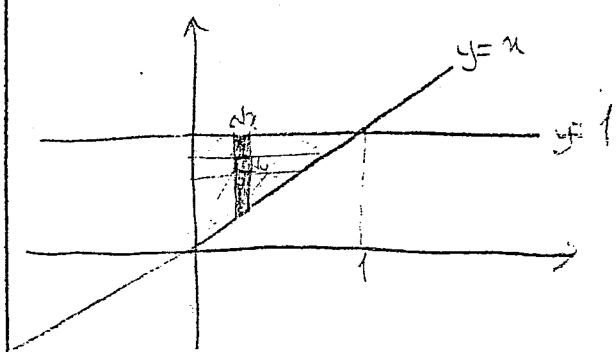
Sıra değiştirme



$\sqrt{x}=x$
$x=x^2$
$0=x^2-x$
$0=x(x-1)$

$$\int_0^1 \left(\int_{y^2}^{y} f(x,y)\,dx \right) dy$$

Örnek: $\int_0^1 \left(\int_x^1 x \cos y^3\,dy \right) dx = ?$



$$\int_0^1 \left(\int_x^1 x \cos y^3\,dy \right) dx = \int_0^1 \left(\int_0^y x \cos y^3\,dx \right) dy$$

$$= \frac{1}{2} \int_0^1 \left| x^2 \cos y^3 \right|_0^y dy$$

$$= \frac{1}{2 \cdot 3} \int_0^1 3y^2 \cdot \cos y^3\,dy$$

$y^3 = t$
$3y^2\,dy = dt$

$$= \frac{1}{6} \int \cos t\,dt$$

$$= \frac{1}{6} \left| \sin y^3 \right|_0^1$$

$= \frac{1}{6}(\sin 1)$

<u>Örnek:</u> $\int_0^1 \left(\int_x^1 e^{y^2} dy \right) dx = ?$

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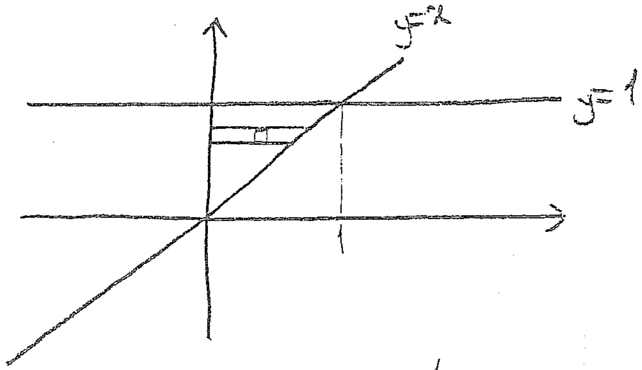


$$\int_0^1 \left(\int_x^1 e^{y^2} dy \right) dx = \int_0^1 \left(\int_0^y e^{y^2} dx \right) dy$$

$$= \int_0^1 \left| e^{y^2} x \right|_0^y dy$$

$$= \frac{1}{2} \int_0^1 2y.e^{y^2} dy \qquad \begin{array}{l} y^2 = t \\ 2ydy = dt \end{array}$$

$$= \frac{1}{2} \int e^t dt$$

$$= \frac{1}{2} \left| e^{y^2} \right|_0^1$$

$$= \frac{1}{2}(e-1)$$

<u>Örnek:</u> $\int_1^{e^2} \left(\int_0^{\ln x} 2x dy \right) dx = ?$

$$\int_1^{e^2} \left(\int_0^{\ln x} 2x dy \right) dx = \int_1^{e^2} 2x \left| y \right|_0^{\ln x} dx$$

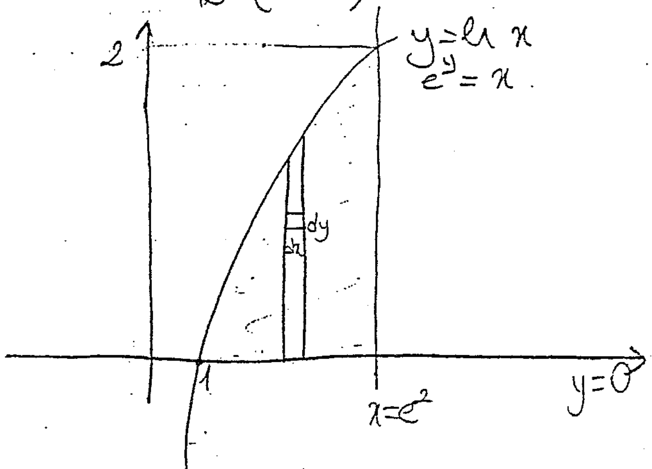
$$\begin{array}{l} \ln x = u \\ \frac{1}{x} dx = du \\ \int x dx = \int dv \\ \frac{x^2}{2} = v \end{array}$$

$$= 2 \int_1^{e^2} x.\ln x dx$$

$$= 2 \left[\frac{x^2}{2} . \ln x \Big|_1^{e^2} - \frac{1}{2} \int_1^{e^2} x^2 . \frac{1}{x} dx \right]$$

$$= x^2 \ln x \Big]_1^{e^2} - \frac{1}{2} x^2 \Big]_1^{e^2}$$

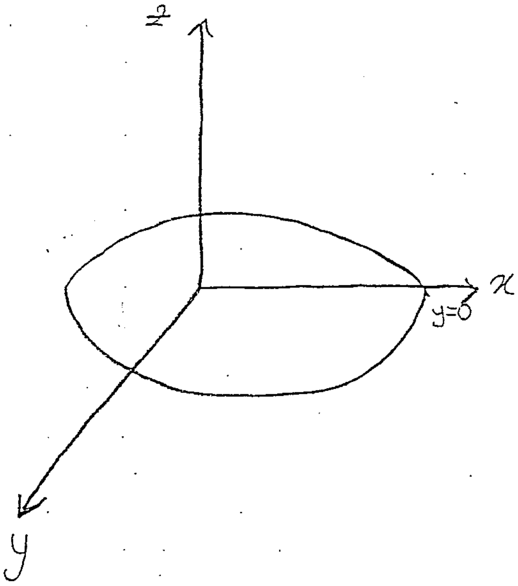
$$= 2e^4 - \frac{1}{2}\left(e^4 - 1\right)$$

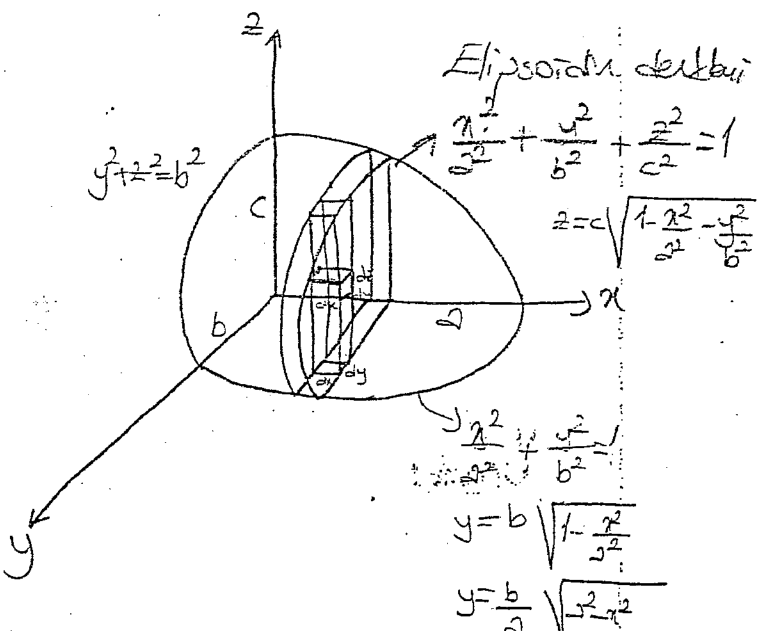


$$\int_0^2 \left(\int_{e^y}^{e^2} (2x)\,dx \right) dy = \int_0^2 x^2 \Big]_{e^y}^{e^2} dy = \int_0^2 (e^4 - e^{2y})\,dy = e^4 y - \frac{1}{2} e^{2y} \Big]_0^2 = 2e^4 - \frac{1}{2}e^4 + \frac{1}{2}$$

$$= \frac{3}{2} e^4 + \frac{1}{2}$$

<u>Örnek:</u> (Elipsoidin hacmi)





Elips döndüğünde elipsoid meydana gelir.

$$\int_0^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \int_{0=z}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} = z} dz\,dy\,dx = V\,br^3$$

$$= c\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\,dx$$

$1 - \frac{x^2}{a^2} = A^2$ $\quad a^2 - x^2 = a^2 A^2$

$\frac{a^2 - x^2}{a^2} = A^2$ $\quad \sqrt{a^2 - x^2} = aA$

$$= c\int_0^a \left(\int_0^{bA} \sqrt{A^2 - \frac{y^2}{b^2}}\, dy \right) dx$$

$\int_0^{bA}$ $y = bA\sin\theta$

$dy = bA\cos\theta\, d\theta$

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$$= c\int_0^a \left[bA^2 \int_0^{\pi/2} \cos^2\theta\, d\theta \right] dx$$

$$= \frac{\pi}{4} cb \int_0^a A^2\, dx$$

$$= \frac{\pi}{4} cb \int_0^a \frac{a^2-x^2}{a^2}\, dx$$

$$= \frac{\pi cb}{4a^2} \int_0^a (a^2-x^2)\, dx$$

$$= \frac{\pi cb}{4a^2} \left| a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= \frac{\pi cb}{\underset{2}{\cancel{4}}\cancel{a^2}} \cdot \frac{\cancel{2}a^{\cancel{3}}}{3}$$

$$= \frac{\pi abc}{6}$$

$$E_{Tam} = \frac{4}{3}\pi abc$$

$$\iint \left[\overset{z_2}{f_2(x,y)} - \overset{z_1}{f_1(x,y)} \right] dy\,dx \longrightarrow \text{Hacim}$$

İKİ KATLI İNTEGRALİN ÖZELLİKLERİ

Teorem:1-) $\iint \left[f(x,y) + g(x,y) \right] dy\,dx = \iint f(x,y)\,dy\,dx + \iint g(x,y)\,dy\,dx$

Teorem:2) a sbt olmak üzere

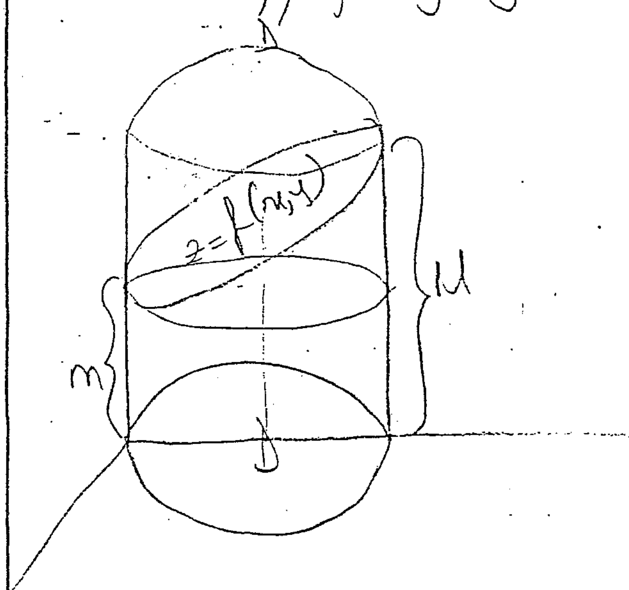
$$\iint a f(x,y)\,dy\,dx = a \iint f(x,y)\,dy\,dx$$

Teorem:3-) D bölgesi kendi içinde $D_1, D_2, \dots, D_n$ gibi alt bölgelere ayrılacak olursa bölgenin tamamında yapılacak integral bu alt bölgelerdeki integrallerin toplamına eşittir.

$$\iint_D f(x,y)\,dy\,dx = \iint_{D_1} f(x,y)\,dy\,dx + \iint_{D_2} f(x,y)\,dy\,dx + \dots + \iint_{D_n} f(x,y)\,dy\,dx$$

Teorem:4-) D bölgesinde m; f(x,y)'nin aldığı en küçük değer, D bölgesinde M; f(x,y)'nin aldığı en büyük değer, A; D bölgesinin alanı olmak üzere

$$m.A \le \iint_D f(x,y)\,dy\,dx \le M.A$$



Teorem:5-) $A.f(P) = \iint f(x,y)\,dy\,dx$

$$f(P) = \frac{1}{A}\iint f(x,y)\,dy\,dx$$

f(P)'ye D bölgesi içindeki fonksiyonun ortalama değeri denir.

I. İhtar: $\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$ ifadesinde parantez kaldırılır

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

II. İhtari

$$\int_{x=a}^{x=b}\int_{x=c}^{x=d}$$

İki integralin de sınırları sabit ise ve $f(x,y)$ fonksiyonu x ve y cinsinden ayrılabiliyorsa

$$\int_{x=a}^{x=b}\int_{x=c}^{x=d} f(x).g(y)\,dy\,dx = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$$ yazılabilir.

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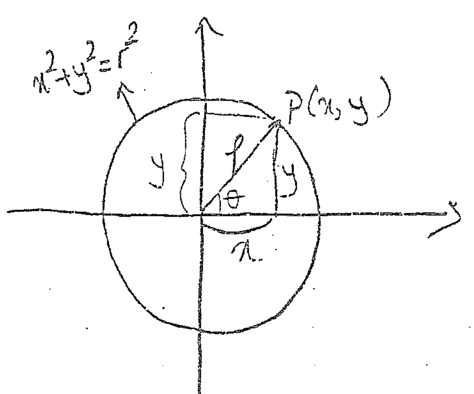
Örnek: $$\int_2^4\int_1^3 xy\,dx\,dy = \int_2^4 y\,dy \int_1^3 x\,dx$$

$$= \frac{y^2}{2}\Big|_2^4 \cdot \frac{x^2}{2}\Big|_1^3$$

$$= 24$$

DÖNÜŞÜMLER

* $\displaystyle\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$ çözümünde zorlanıyorsanız ve D bölgesi dairesel bir bölge (çember) ise kutupsal koordinatlar uygun bir çözümdür.



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx = \iint_{D_{kut}} \varphi(\rho,\theta)\,|J|\,d\rho\,d\theta$$

$$J = \frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} x_\rho' & x_\theta' \\ y_\rho' & y_\theta' \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$$

Kutupsal koordinatlarda;

$$\iint f(x,y)\,dx\,dy = \iint \varphi(\rho,\theta)\,\rho\,d\rho\,d\theta$$

Silindirik koordinatlarda da durum aynıdır.

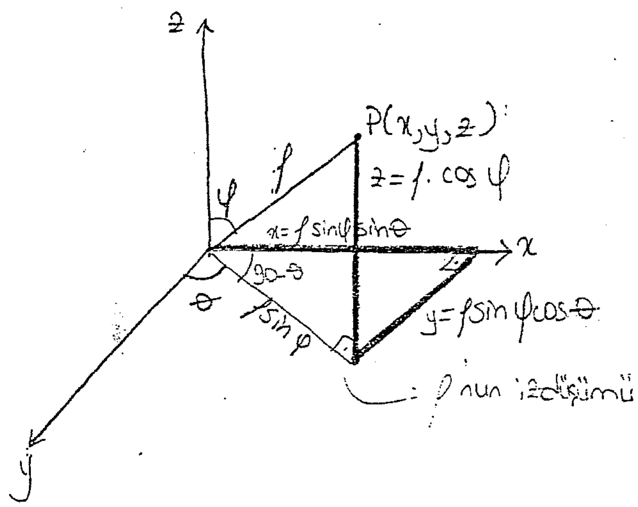
$x^2+y^2=r^2$ de Silindir denklemidir.

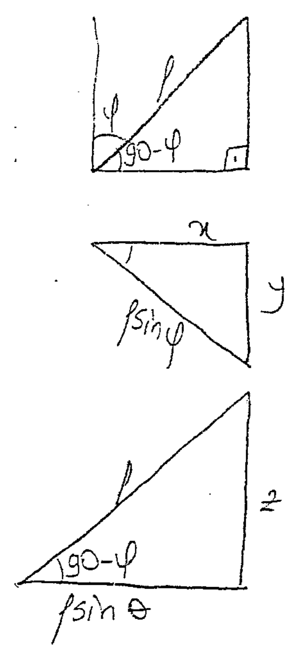
$x^2+y^2+z^2=r^2$ Kürenin denklemi

Kartezyen koordinatlardan kutupsal koordinatlara geçtiğimizde $dx\,dy$'nin karşılığı $\rho\,d\rho\,d\theta$'dır.

$dx\,dy \rightarrow \rho\,d\rho\,d\theta$

* Küresel Koordinatlar:





$x = \rho.\sin\varphi.\sin\theta$

$y = \rho.\sin\varphi.\cos\theta$

$z = \rho.\cos\varphi$

$$\iiint dx\,dy\,dz = \iiint |J|\,d\rho\,d\theta\,d\varphi$$

$$\frac{D(x,y,z)}{D(\rho,\theta,\varphi)} = \begin{vmatrix} x'_\rho & x'_\theta & x'_\varphi \\ y'_\rho & y'_\theta & y'_\varphi \\ z'_\rho & z'_\theta & z'_\varphi \end{vmatrix} = \begin{vmatrix} \sin\varphi.\sin\theta & \rho\sin\theta\cos\theta & \rho\cos\varphi.\sin\theta \\ \sin\varphi.\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi.\cos\theta \\ \cos\varphi & 0 & -\rho.\sin\varphi \end{vmatrix}$$

$$= \cos\varphi\left(\rho^2\sin\varphi\cos\varphi.\cos^2\theta + \rho^2\sin\varphi\cos\varphi\sin^2\theta\right) - \rho\sin\varphi\left(-\rho\sin^2\varphi\sin^2\theta - \rho\sin^2\varphi\cos^2\theta\right)$$

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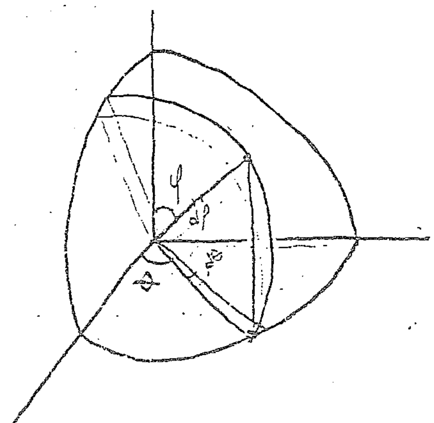
$$= \rho^2\sin\varphi\cos^2\varphi + \rho^2\sin^3\varphi$$

$$= \rho^2\sin\varphi$$

$$\iiint dx.dy dz = \iiint \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$



ρ nokt $\int_0^r$ $\theta \int_0^{\pi/2}$ $\varphi \int_0^{\pi/2}$

$$\iiint dx dy dz = \int_0^r \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \sin\varphi\, d\varphi\, d\theta\, d\rho$$

$$= \frac{\pi}{2} \int_0^r \int_0^{\pi/2} \rho^2 . d\rho \sin\varphi\, d\varphi$$

$$= \frac{\pi r^3}{2.3} \int_0^{\pi/2} \sin\varphi\, d\varphi$$

$$= \frac{\pi r^3}{6} \quad (8'de\ biri)$$

$$8.^4 \frac{\pi r^3}{6_3} = \frac{4}{3}\pi r^3$$

Kutupsalda dönüştürmek için yüzey integrali olmalıdır

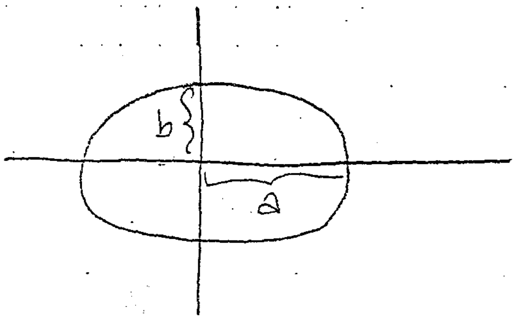
$$\iiint dx\,dy\,dz = \iint \overset{f(x,y)}{z}\,dx\,dy$$

$$\Downarrow$$

$$\iint f(x,y)\,\rho\,d\rho\,d\theta$$

$$\iiint \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

*** Eliptik Koordinatları**



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow u^2 + v^2 = 1$$

$$\left.\begin{aligned} \frac{x}{a} &= u \\ \frac{y}{b} &= v \end{aligned}\right\}$$

$$u = \rho\cos\theta$$
$$v = \rho\sin\theta$$

$$\frac{x}{a} = \rho\cos\theta \Rightarrow x = a\,\rho\cos\theta$$

$$\frac{y}{b} = \rho\sin\theta \Rightarrow y = b\,\rho\sin\theta$$

$$\iint_K f(x,y)\,dx\,dy \overset{\text{eliptik}}{=} \iint_E p(\rho,\theta)\,|J|\,d\rho\,d\theta$$

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} x'_\rho & x'_\theta \\ y'_\rho & y'_\theta \end{vmatrix} = \begin{vmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{vmatrix} = a\,b\,\rho$$

$$\iint_K f(x,y)\,dx\,dy = \iint_E p(\rho,\theta)\,a\,b\,\rho\,d\rho\,d\theta$$

*** Silindirik Koordinatları**

$$x^2 + y^2 = r^2$$

Kar → K

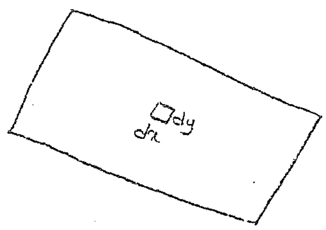
$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$
$$z = z$$

Kar. → Kut.	$\lvert J\rvert = \rho$
Kar. → Sil.	$\lvert J\rvert = \rho$
Kar. → Elip.	$\lvert J\rvert = ab\rho$
Kar. → Kür.	$\lvert J\rvert = \rho^2 \sin\varphi$

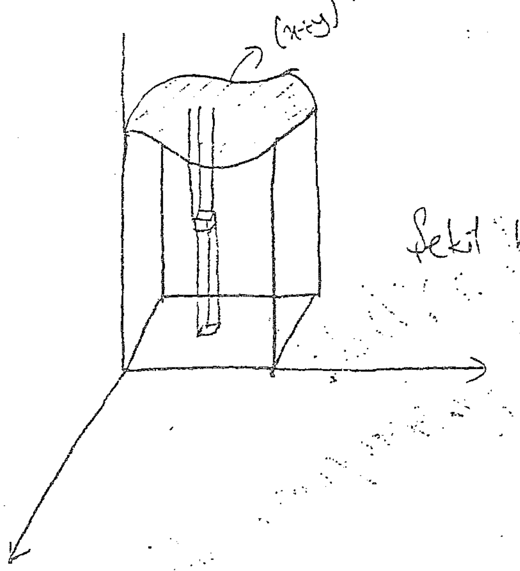
Örnek: D bölgesi (1,0), (2,2), (1,3), (0,1) noktalarını köşe kabul eden bir paralelkenar olsun.

$$\iint_D (x+y)^4\,dx\,dy$$ integre ediniz.

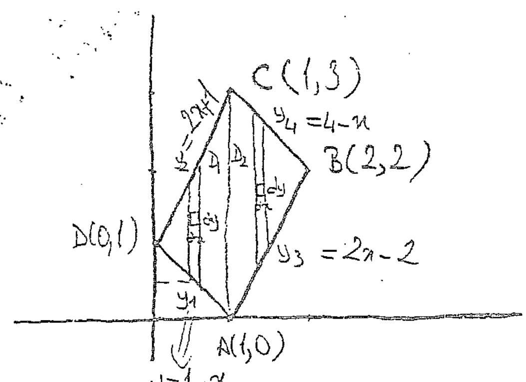
Kesiti alanı dy, yüksekliği $(x+y)^4$ olan ucurtma gibi



$$\iiint_0^{(x+y)^4} dz\,dy\,dx$$



Şekil böyle değil.



$$\iint (x+y)^4\,dx\,dy = \iint_{D_1} (x+y)^4\,dy\,dx + \iint_{D_2} (x+y)^4\,dy\,dx$$

Eksenleri kesen doğru denklemi: $\frac{x}{a}+\frac{y}{b}=1$

$\frac{x}{1}+\frac{y}{1}=1 \Rightarrow y=1-x$

İki noktası belli olan doğru denklemi: $\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$

$\frac{y-1}{1-3}=\frac{x-0}{0-1} \Rightarrow -y+1=-2x \Rightarrow y=2x+1$

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$$\frac{y-3}{3-2} = \frac{x-1}{1-2} \Rightarrow -y+3 = x-1 \Rightarrow y = 4-x$$

$$\frac{y-2}{2-0} = \frac{x-2}{2-1} \Rightarrow y-2 = 2x-4$$
$$y = 2x-2$$

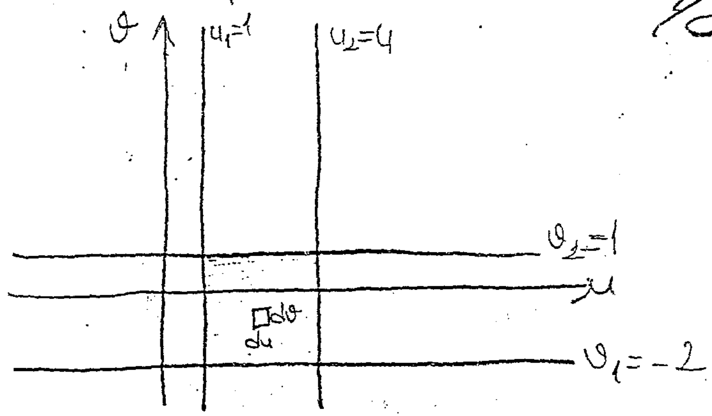
$x+y=1$
$x+y=4$
$-2x+y=1$
$-2x+y=-2$

$$\iint (x+y)^4 \, dx dy = \iint_{D_1} (x+1)^4 \, dy dx + \iint_{D_2} (x+1)^4 \, dy dx$$

$$= \int_0^1 \int_{1-x}^{2x+1} (x+y)^4 \, dy dx + \int_1^2 \int_{2x-2}^{4-x} (x+y)^4 \, dy dx$$

$x+y=u \begin{cases} u_2 = 4 \\ u_1 = 1 \end{cases}$

$-2x+y=v \begin{cases} v_2 = 1 \\ v_1 = -2 \end{cases}$



Meriba!
Yazmıyan di mi?!...

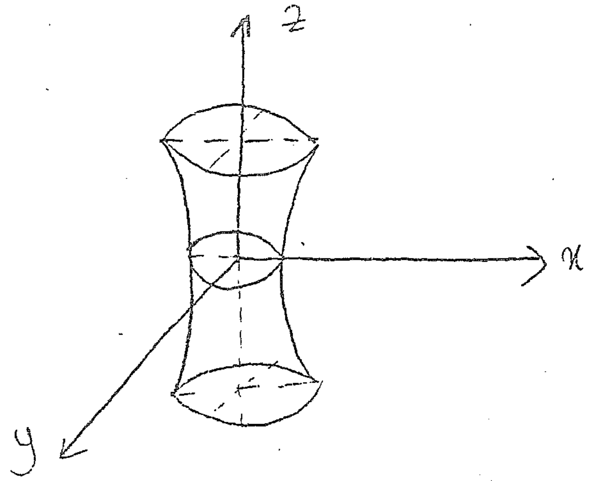
$$\int_1^3 \int_3^7 \int_2^4 \int_6^8 dx dy dz dt = \int_1^3 dt \cdot \int_3^7 dz \int_2^4 dy \int_6^8 dx$$

$$\iint u^4 \, |J| \, dv du = \int_1^4 \int_{-2}^1 u^4 \, \frac{1}{3} \, dv du = \frac{1}{3} \int_1^4 u^4 du \int_{-2}^1 dv = \frac{1}{3} \left| \frac{u^5}{5} \right|_1^4 \cdot \left| v \right|_{-2}^1$$

$$= \frac{1}{\cancel{3}} \left[\frac{1024}{5} - \frac{1}{5} \right] \cdot \cancel{\left[1+2 \right]}$$

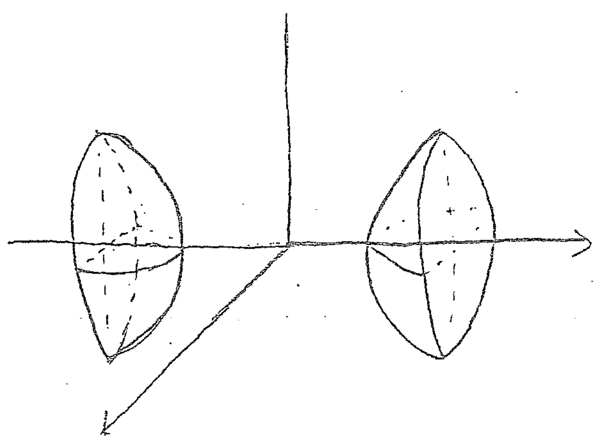
$$= \frac{1023}{5} \text{ br}^3$$

* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Bir kollu hiperboloid



a=b ise kesiti çember olan tek kollu hiperboloid
a≠b " " elips " " " "

* İki Kollu Hiperboloid

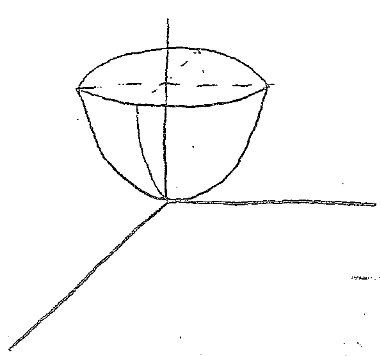


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

* Paraboloid

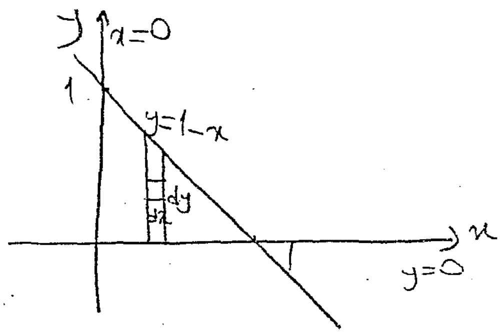
$$2cz=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

a≠b ise kesiti eliptik olan paraboloid
a=b " " çember " "

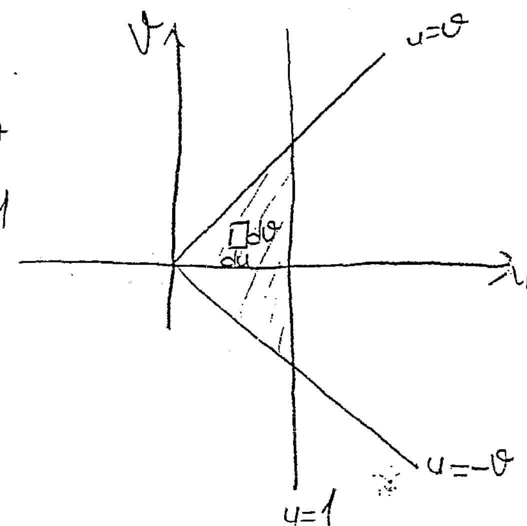


$$\frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}} = \frac{1}{\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{1}{3}$$

Örnek: $\int_0^1 \int_0^{1-x} e^{y-x/y+x} \, dy\,dx$



$y+x=u$, $y-x=v$ $\begin{cases} x=0 & u=v \\ y=0 & u=-v \\ y+x=1 & u=1 \end{cases}$



$$\iint e^{v/u} |J| \, dv\,du = \frac{1}{2} \int_0^1 \int_{-u}^{u} e^{v/u} \, dv\,du = \frac{1}{2} \int_0^1 u.e^{v/u} \Big|_{-u}^{u} du = \frac{1}{2} \left\{ \int_0^1 u(e-e^{-1})du \right\}$$

$$\frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}} = \frac{1}{\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{1}{2}$$

$$= \frac{1}{2} (e-e^{-1}) \frac{u^2}{2} \Big|_0^1$$

$$= \frac{1}{4} (e-e^{-1})$$

$$= \frac{1}{2} \, sh\, 1 \quad br^3$$

$e^{\frac{2}{5}dx} = e^{\frac{1}{5}x} dx$

$e^{\frac{1}{u}.v} = u.e^{v/u}$

İKİNCİ DERECEDEN YÜZEYLER

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+K=0$$

* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ise elipsoiddir.

$c=b$ ise kesiti dairesel olan, $b \neq c$ ise kesiti elips olan elipsoid

$\frac{(x-a_1)^2}{a^2} + \frac{(y-b_1)^2}{b^2} + \frac{(z-c_1)^2}{c_2} = 1$ Merkezel olmayan

* $a=b=c=r$ ise küredir.

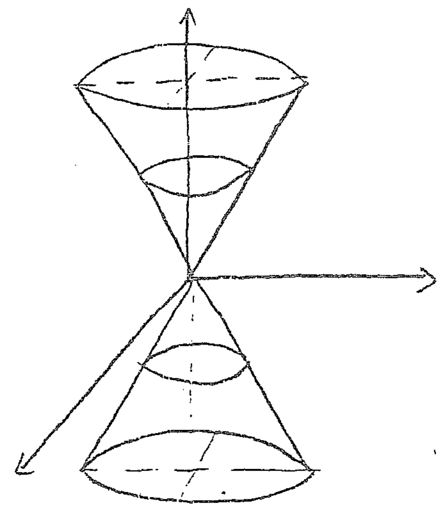
$x^2+y^2+z^2=r^2$

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

* Hiperbolik Paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2cz$$

* Dik Çember Konisi



$$x^2+y^2-c^2z^2=0$$

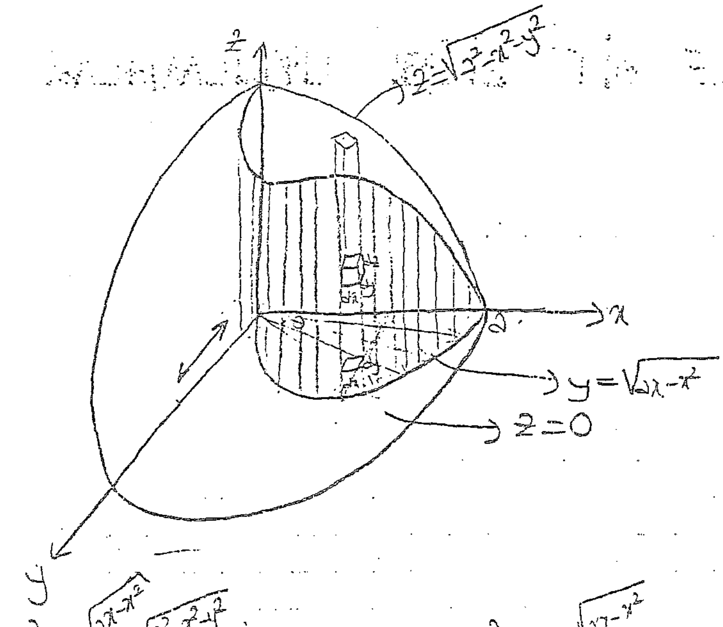
* Silindir

$x^2+y^2=r^2$ Kesiti çember olan silindir

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Kesiti elips olan silindir

Örnek: Viviani bacası

$x^2+y^2+z^2=a^2$, $x^2+y^2-ax=0$ yüzeylerinin sınırl dığı hacmi bulun



$$x^2-ax+\frac{a^2}{4}-\frac{a^2}{4}+y^2=1$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

$$4\int_0^a\int_0^{\sqrt{ax-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 4\int_0^a\int_0^{\sqrt{ax-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx$$

Kutupsala dönüştür

$$= 4 \int_0 \int_0 \sqrt{a^2-\rho^2} \cdot \rho\, d\rho\, d\theta$$

$x=\rho\cos\theta$
$y=\rho\sin\theta$

$x^2+y^2-2x=0$ 78

$\rho^2-2\rho\cos\theta=0$
$\rho(\rho-2\cos\theta)=0$

Ariçap çemberin denklemine hareket eder.

$$= 4 \int_0^{\pi/2} \left[\int_0^{2\cos\theta} \left(\sqrt{a^2-\rho^2}\, \rho\, d\rho\right) \right] d\theta$$

$$= -4 \int_0^{\pi/2} \left(\int t^2 dt \right) d\theta$$

$a^2-\rho^2=t^2$
$-\cancel{2}\rho\, d\rho=\cancel{2}t\, dt$
$\rho\, d\rho=-t\,dt$

$$= -\frac{4}{3} \int_0^{\pi/2} t^3 \cdot d\theta$$

$$= -\frac{4}{3} \int_0^{\pi/2} \left| (a^2-\rho^2)^{3/2} \right|_0^{a\cos\theta} d\theta$$

$$= -\frac{4}{3} \int_0^{\pi/2} \left[\left(a^2-a^2\cos^2\theta\right)^{3/2} - (a^2)^{3/2} \right] d\theta$$

$$= -\frac{4}{3} \left[\int_0^{\pi/2} \left(a^3\sin^3\theta - a^3 \right) d\theta \right]$$

$x^2+y^2-ax=0$
$\rho^2-a\cdot\rho\cos\theta=0$
$\rho(\rho-a\cos\theta)=0$

$$= -\frac{4a^3}{3} \int_0^{\pi/2} \sin^3\theta\, d\theta + \frac{4a^3}{3} \int_0^{\pi/2} d\theta$$

$$= -\frac{4a^3}{3} \cdot \frac{2}{3} + \frac{4a^3}{3} \cdot \frac{\pi}{2}$$

$$= \frac{4a^3}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right) br^3$$

ÜÇ KATLI İNTEGRALE AİT DİĞER UYGULAMALAR

1-) Yoğunluğu $f(x,y,z)$ olan bir cismin kütlesi

$$M = \iiint_V f(x,y,z)\, dz\, dy\, dx \text{ 'tir.}$$

2-) Bir cismin eylemsizlik momenti

OX eksenine göre $$I_{ox} = \iiint_V (y^2+z^2)\, f(x,y,z)\, dx\, dy\, dz$$

$$I_{oy} = \iiint_V (x^2+z^2)\, f(x,y,z)\, dx\, dy\, dz$$

$$I_{oz} = \iiint_V (x^2+y^2)\, f(x,y,z)\, dx\, dy\, dz$$

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$$I_{xoy} = \iiint_V z^2 \, f(x,y,z)\,dx\,dy\,dz$$

$$I_{yoz} = \iiint_V x^2 \, f(x,y,z)\,dx\,dy\,dz$$

$$I_{xoz} = \iiint_V y^2 \, f(x,y,z)\,dx\,dy\,dz$$

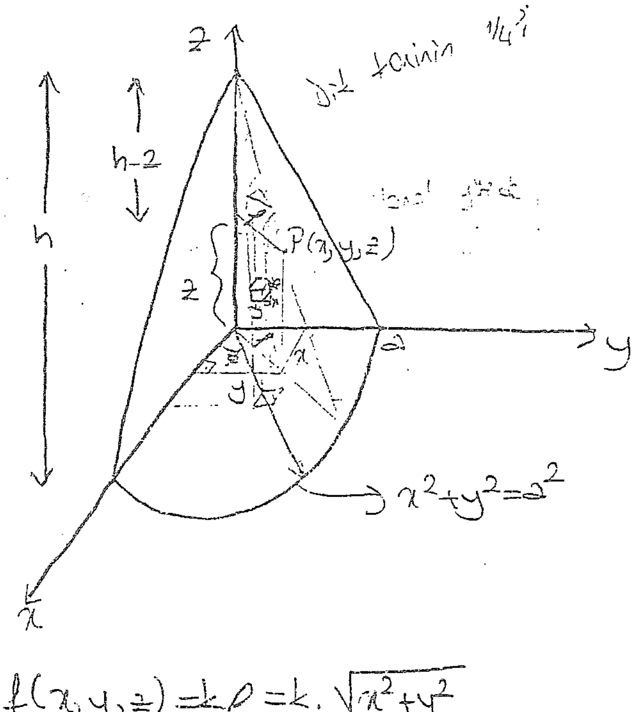
3-) Bir cismin ağırlık merkezi $G(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{\iiint x \cdot f(x,y,z)\,dx\,dy\,dz}{\iiint f(x,y,z)\,dx\,dy\,dz}$$

$$\bar{y} = \frac{\iiint y \cdot f(x,y,z)\,dx\,dy\,dz}{M}$$

$$\bar{z} = \frac{\iiint z \cdot f(x,y,z)\,dx\,dy\,dz}{M}$$

Örnek: Herhangi bir noktasındaki yoğunluğu eksene uzaklığı ile orantılı olan bir dik koninin kütlesini bulunuz.



$\rho \to$ Eksene olan uzaklık

$$f(x,y,z) = k\rho = k \cdot \sqrt{x^2+y^2}$$

z=0 ile $x^2+y^2=a^2$ dir

$$\frac{\rho}{a} = \frac{h-z}{h}$$

$$\frac{\rho h}{a} = h - z \implies z = h\left(1 - \frac{\rho}{a}\right) \implies z = h\left(1 - \frac{\sqrt{x^2+y^2}}{a}\right) \Rightarrow \text{Koni denklem}$$

$$M = \iiint_V f(x,y,z)\,dz\,dy\,dx$$

(h = boyunu koninin denklemine göre belirle

$$= 4k\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{h\left(1-\frac{\sqrt{x^2+y^2}}{a}\right)} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$= 4kh\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\left(1-\frac{\sqrt{x^2+y^2}}{a}\right)dy\,dx$$

$$= 4kh\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{x^2+y^2} - \frac{x^2+y^2}{a}\right]dy\,dx$$

dydx cöresel bölgesi olduğu için kutupsal koordinatlara dönüştürebiliriz

$$= 4kh\int_0^a \int_0^{\pi/2} \left(\rho - \frac{\rho^2}{a}\right)\rho\,d\theta\,d\rho$$

$$= 2\pi kh\int_0^a \left(\rho^2 - \frac{\rho^3}{a}\right)d\rho$$

$$= 2\pi kh \left| \frac{\rho^3}{3} - \frac{\rho^4}{4a} \right|_0^a$$

$$= \frac{a^3\pi kh}{6}$$

Kitabın Çözüm Yolu:

$$M = k.\int_0^h \int_0^{2\pi} \int_0^{\frac{a}{h}(h-z)} \rho^2\,d\rho\,d\theta\,dz$$

$$\iiint k\rho\,dx\,dy\,dz$$

kutupsala dönüştürülmüş

$$k\rho.\rho\,d\rho\,d\theta\,dz$$

$$\frac{\rho}{a} = \frac{h-z}{h} \rightarrow \rho = \frac{a}{h}(h-z)$$

$$= k\int_0^h \int_0^{2\pi} \frac{\rho^3}{3}\Bigg|_0^{\frac{a}{h}(h-z)} d\theta\,dz$$

$$= \frac{k}{3}\int_0^h \int_0^{2\pi} \frac{a^3}{h^3}(h-z)^3\,d\theta\,dz$$

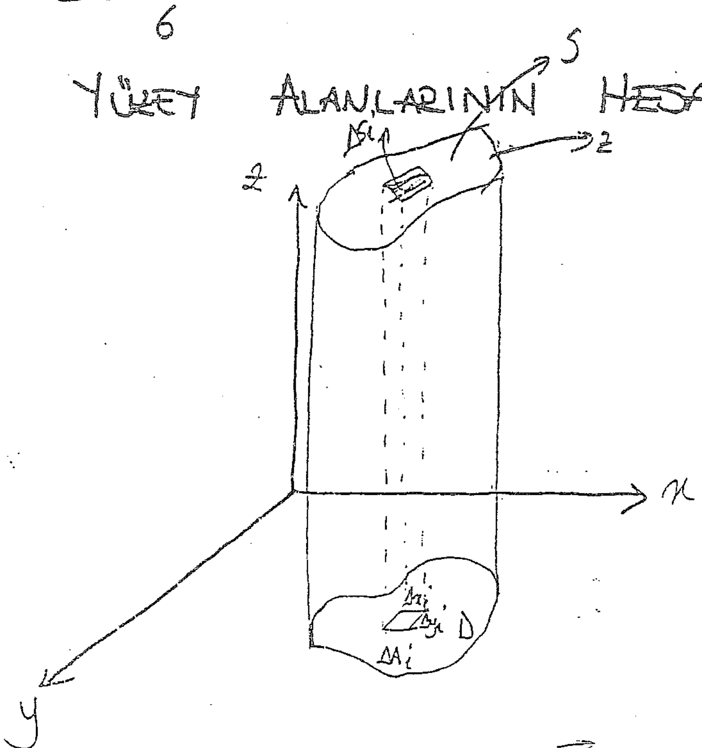
$$= \frac{ka^3 2\pi}{3h^3}\int_0^h (h-z)^3\,dz$$

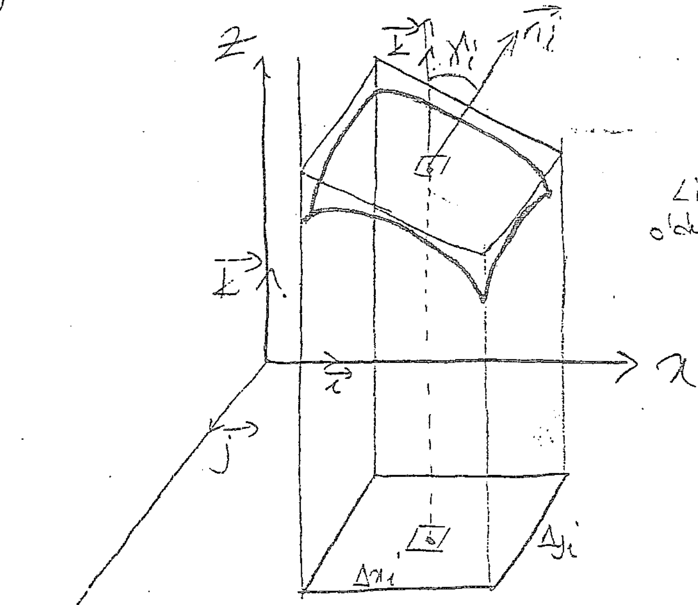
$$= -\frac{2ka^3\pi}{3h^3}\left| \frac{(h-z)^4}{4} \right|_0^h$$

$$= \frac{-\cancel{2}ka^3\pi}{3h^3}\left(-\frac{h^4}{\cancel{4}_2}\right)$$

$$= \frac{\pi k a^3 h}{6}$$

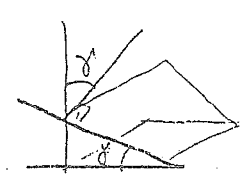
YÜZEY ALANLARININ HESABI





ΔA_i

Limit durumunda iki alanın eşit olduğu kabul edilir.



$$\Delta A_i = \Delta S_i \cdot \cos\gamma_i$$

$$\Delta A_i = \Delta x_i \, \Delta y_i$$

$$\Delta A_i = \Delta S_i \cdot \cos\gamma_i$$

$$\vec n_i \cdot \vec k = |\vec n_i| \cdot \cos\gamma_i$$

$$\left(-\frac{\partial z}{\partial x}\vec i - \frac{\partial z}{\partial y}\vec j + \vec k\right)\cdot \vec k = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \cdot 1 \cdot \cos\gamma_i$$

$$1 = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \cdot \cos\gamma_i$$

$$|\vec a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

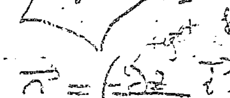
$$\vec a = a_1 \vec i + a_2 \vec j + a_3 \vec k$$

$$\vec b = b_1 \vec i + b_2 \vec j + b_3 \vec k$$

$$\vec a \cdot \vec b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec a \cdot \vec b = a.b.\cos\theta$$

Uzayda herhangi bir yüzeyin normalini bulmak için;



geniş açı yapar

$$\vec n = \left(-\frac{\partial z}{\partial x}\vec i - \frac{\partial z}{\partial y}\vec j + \vec k\right)$$

$$\Delta A_i = \Delta S_i \cdot \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}}$$

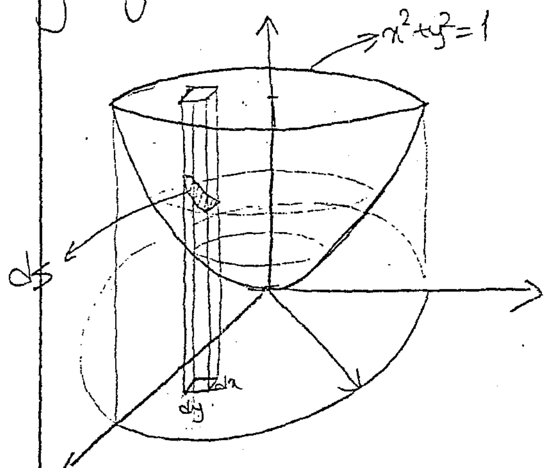
$$\Delta S_i = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} \cdot \Delta A_i$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta A_i = dA$$

$$\int ds = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy \qquad x^2+y^2=1=\rho^2$$

Örnek: $z=x^2+y^2$ paraboloidinin $z=1$ düzleminin altında kalan yüzeyinin alanını bulunuz.



$$\frac{\partial z}{\partial x}=2x \qquad \frac{\partial z}{\partial y}=2y$$

$$S=\iint \sqrt{1+4x^2+4y^2}\, dy\,dx$$

$$=\int_0^1\int_0^{2\pi} \sqrt{1+4\rho^2}\, \rho\, d\rho\, d\theta$$

$$=2\pi\int_0^1 \sqrt{1+4\rho^2}\, \rho\, d\rho \qquad \begin{aligned} 1+4\rho^2&=t^2 \\ 8\rho\, d\rho&=2t\,dt \\ 4\rho\, d\rho&=t\,dt \end{aligned}$$

$$=\frac{2\pi}{4}\int t\cdot t\,dt$$

$$=\frac{\pi}{2}\left|\frac{t^3}{3}\right|$$

$$=\frac{\pi}{6}\left|\left(1+4t^2\right)^{3/2}\right|_0^1$$

$$=\frac{\pi}{6}\left(5^{3/2}-1^{3/2}\right)\, br^2$$

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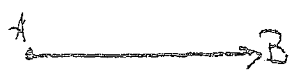
VEKTÖRLER

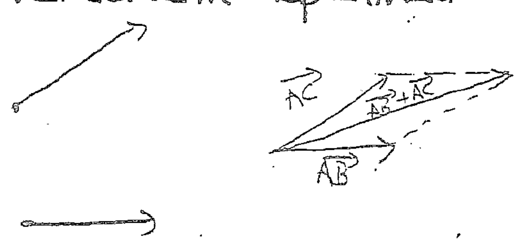
Başlangıç noktası, doğrultusu, yönü, şiddeti belli olan büyüklüklere vektör denir.

a) Sabit vektör
b) Kayan vektör
c) Serbest vektör



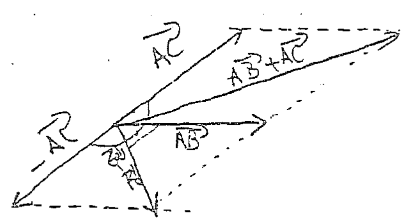
$\vec{AB} = -\vec{BA}$

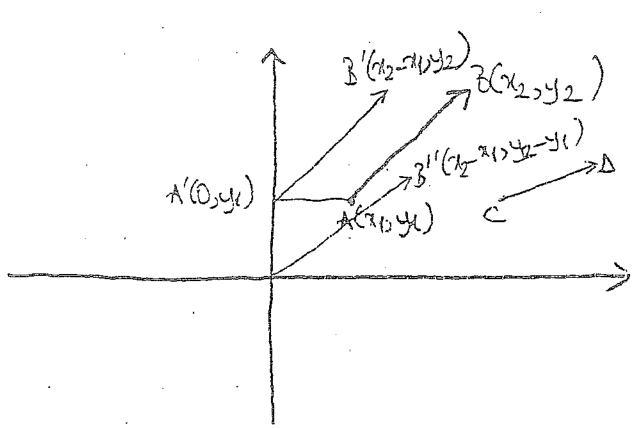
Vektörlerin Toplanması:



Vektörlerin Çıkarılması:

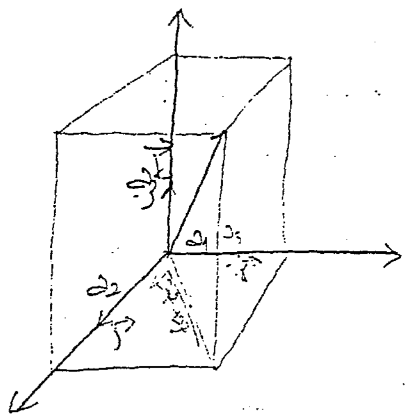
$\vec{AB}+(-\vec{AC})=\vec{AB}-\vec{AC}$





Başlangıçları orjine gelen vektörlere yer vektörü denir.

$\vec{AB} = (x_2 - x_1 , y_2 - y_1)$ Yer vektörü
$\vec{a} = (a_1 , a_2)$



$|\vec{a}| \to \vec{a}$ vektörünün şiddeti

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ Herhangi bir vektörün birim vektör olarak ifadesi

$\vec{i}, \vec{j}, \vec{k}$ ya baz birim vektörler denir.

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1, a_2, a_3)$
$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = (b_1, b_2, b_3)$

A(1,2,3) uzaydaki bir nokta
$\vec{A} = (1, 2, 3) = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{a} \mp \vec{b} = (a_1 \mp b_1)\vec{i} + (a_2 \mp b_2)\vec{j} + (a_3 \mp b_3)\vec{k} = [a_1 \mp b_1 , a_2 \mp b_2 , a_3 \mp b_3]$

* Bir vektörü skalerle çarpmak vektörü o kadar büyütmek demektir.

$3\vec{a}$, $\frac{1}{2}\vec{b}$

Ö5/ A(2,3,1) B(-1,0,2) C(4,0,-1) D(1,1,0)

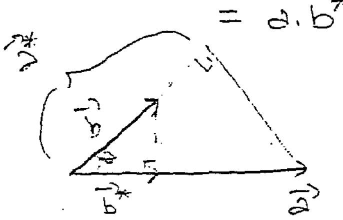
$\vec{AB} + \vec{CD} = (-3,-3,1) + (-3,1,1)$
$= (-6,-2,2)$

$3\vec{AB} + 4\vec{CD} = 3(-3,-3,1) + 4(-3,1,1)$
$= (-21,-5,7)$

VEKTÖRLERİN ÇARPIMI

1-) Skaler Çarpım:

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|.\cos\theta$
$= a.b^* = b.a^*$



$b^* = b\cos\theta$

$a^* = a\cos\theta$

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

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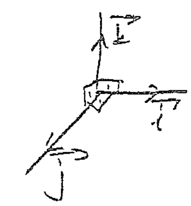
$\vec{a}\cdot\vec{b} = a_1b_1\,\vec{i}\cdot\vec{i} + a_1b_2\,\vec{i}\cdot\vec{j} + a_1b_3\,\vec{i}\cdot\vec{k} + a_2b_1\,\vec{j}\cdot\vec{i} + a_2b_2\,\vec{j}\cdot\vec{j} + a_2b_3\,\vec{j}\cdot\vec{k} +$
$a_3b_1\,\vec{k}\cdot\vec{i} + a_3b_2\,\vec{k}\cdot\vec{j} + a_3b_3\,\vec{k}\cdot\vec{k}$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$



İki vektör arasındaki açı $90°$ ise bu vektörlerin skaler çarpımı 0'dır, yani skaler çarpımları 0 olan iki vektör birbirine diktir.

Örnek: $\vec{a} = (2,-1,4)$
$\vec{b} = (1,0,3)$

$\vec{a}\cdot\vec{b} = 2+0+12 = 14$

$14 = \sqrt{21}\cdot\sqrt{10}\cdot\cos\theta$

$\cos\theta = \dfrac{14}{\sqrt{210}}$

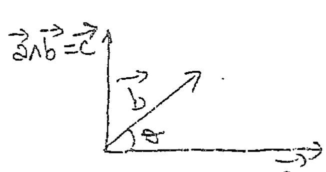
$\theta = \operatorname{Arc}\cos\dfrac{14}{\sqrt{210}}$

Örnek: $\vec{a} = (2,3,6)$
$b^* = 5$

$\vec{a}\cdot\vec{b} = 35$

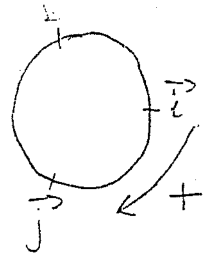
2-) Vektörel Çarpımı

$\vec{a}\wedge\vec{b} = a.b\,\sin\theta.\vec{u}$



$\vec{a}\wedge\vec{b} = a_1b_1\,\vec{i}\wedge\vec{i} + a_1b_2\,\vec{i}\wedge\vec{j} + a_1b_3\,\vec{i}\wedge\vec{k} + a_2b_1\,\vec{j}\wedge\vec{i} + a_2b_2\,\vec{j}\wedge\vec{j} + a_2b_3\,\vec{j}\wedge\vec{k} +$
$a_3b_1\,\vec{k}\wedge\vec{i} + a_3b_2\,\vec{k}\wedge\vec{j} + a_3b_3\,\vec{k}\wedge\vec{k}$

$\vec{i}\wedge\vec{j} = 1.1.1.\vec{k}$
$\vec{i}\wedge\vec{i} = 0$



$$\vec{a}\wedge\vec{b} = a_1b_2\vec{k} - a_1b_3\vec{j} - a_2b_1\vec{k} + a_2b_3\vec{i} + a_3b_1\vec{j} - a_3b_2\vec{i} \ \dots$$

$$= a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} - a_3b_2\vec{i} - a_1b_3\vec{j} - a_2b_1\vec{k}$$

$$\vec{a}\wedge\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a}\wedge\vec{b} = (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Örnek: $\vec{a} = (1,1,0)$
$\vec{b} = (0,-1,1)$

$$\vec{a}\wedge\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \vec{i} - \vec{j} - \vec{k} = \vec{c}$$

$$\vec{u_c} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}$$

$$\vec{u_c} = \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$$

$$\vec{a}\cdot\vec{b} = -1 = 2\cos\theta$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}$$

Rasgele bir $\vec{c}$ vektörünün birim vektörü $\vec{u}$ olsun.

$$\vec{u}_c = \frac{\vec{c}}{|\vec{c}|}$$

10m yürüdüm. Birimim $= \frac{\cancel{10}m}{\cancel{10}}$

İki vektöre dik bir vektör isteniyorsa vektörel çarpımı bul. Birim vektörü bulup istenilen şiddetle çarp.

Skaler Çarpımın Özellikleri:

1-) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
2-) $k(\vec{a}\cdot\vec{b}) = k\vec{a}\cdot\vec{b}$
3-) $\vec{c}\cdot(\vec{a}+\vec{b}) = \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b}$

Vektörel Çarpımın Özellikleri:

1-) $\vec{a}\wedge\vec{b} \neq \vec{b}\wedge\vec{a}$ $\quad \vec{a}\wedge\vec{b} = -\vec{b}\wedge\vec{a}$
2-) Vektörel çarpım bir vida hareketidir.
3-) $\vec{c}\wedge(\vec{a}+\vec{b}) = \vec{c}\wedge\vec{a} + \vec{c}\wedge\vec{b}$
4-) $|\vec{a}\wedge\vec{b}| = a.b.\sin\theta$ paralelkenarın alanı.

$S = \frac{1}{2} ab \sin\theta$ (üçgen alanı)

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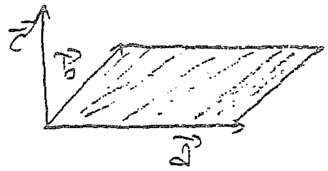


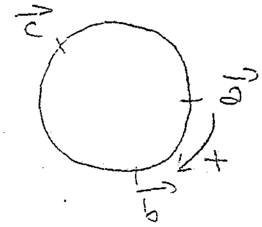
3-) Karma Çarpımı

$\vec{a} \cdot (\vec{b} \wedge \vec{c})$

$\cancel{\vec{a} \wedge (\vec{b} \cdot \vec{c})}$

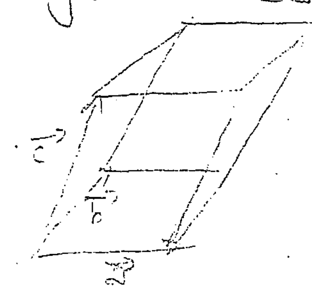
$\vec{a} \cdot (\vec{b} \wedge \vec{c})$

$\vec{a} \cdot \vec{d} = k$



$$\vec{a} \cdot (\vec{b} \wedge \vec{c}) = \vec{b} \cdot (\vec{c} \wedge \vec{a}) = \vec{c} \cdot (\vec{a} \wedge \vec{b})$$

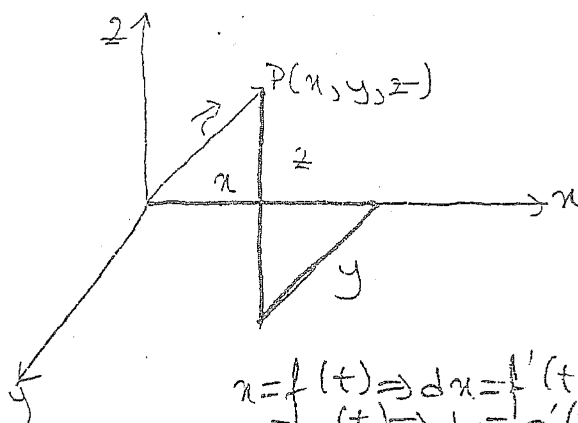
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ Paralel yüzünün hacmi



VEKTÖREL ANALİZ

$f(x,y,z) = 0 \Rightarrow$ Fonksiyonunun geometrik yeri uzayda bir eğriyi (cismi) gösterir.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$



$\vec{r}$: yer vektörü

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \left\{ \begin{array}{l} x = f(t) \\ y = g(t) \\ z = h(t) \end{array} \right\}$$

$$\Rightarrow \vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

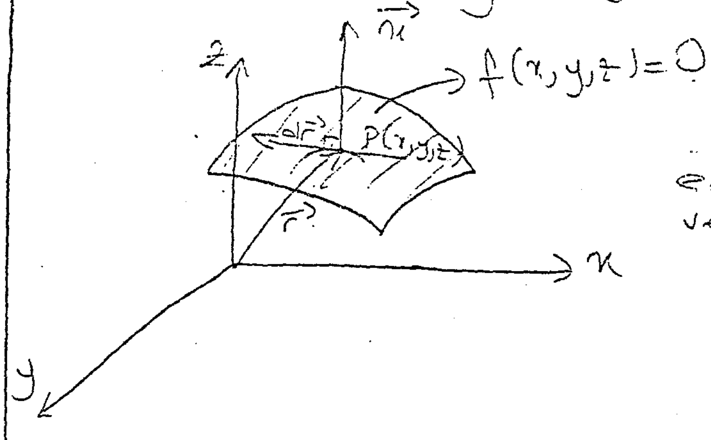
$$d\vec{r} = f'(t)\,dt\,\vec{i} + g'(t)\,dt\,\vec{j} + h'(t)\,dt\,\vec{k}$$

$$\left. \begin{array}{l} x = f(t) \Rightarrow dx = f'(t)dt \\ y = g(t) \Rightarrow dy = g'(t)dt \\ z = h(t) \Rightarrow dz = h'(t)dt \end{array} \right\} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{n} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$\vec{n_i} \cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$



$d\vec{r}$: O noktadan geçen bi eğriye O noktada teğet vektördür.

teğet vektörü $\perp$ normal vek

$d\vec{r} \perp \vec{n_i}$

$\vec{n_i} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\vec{n_i} = \underbrace{\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)}_{\vec{\nabla}\text{ (Nabla vektörü)}} f(x,y,z)$

$\text{Grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ (f yüzeyinin herhangi bir noktasındaki normali) sonuç vektördür.

Gradyantın Özellikleri:

gradyant sonucu vektör

1) f ve g skaler fonksiyonlardır.

$\vec{\nabla}(f \mp g) = \vec{\nabla} f \mp \vec{\nabla} g$

2) Çarpımın gradyantı

$\vec{\nabla}(f \cdot g) = f \cdot \vec{\nabla} g + g \cdot \vec{\nabla} f$

$\left(\frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f\right)\vec{i}$

$+ \left(\frac{\partial f}{\partial y} g + \frac{\partial g}{\partial y} f\right)\vec{j}$

$+ \left(\frac{\partial f}{\partial z} g + \frac{\partial g}{\partial z} f\right)\vec{k}$

$= \underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)}_{\vec{\nabla} f} g + \underbrace{\left(\frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k}\right)}_{\vec{\nabla} g} f$

$= g\vec{\nabla} f + f\vec{\nabla} g$

3-) $\vec{\nabla}(cf) = c\,\vec{\nabla} f$ c=sbt olmak üzere.

Diverjans:

$\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$

Ancak bir vektörün diverjansı alınır.

$\mathrm{Div}\,\vec{V} = \vec{\nabla}\circ\vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\circ\left(V_1\vec{i} + V_2\vec{j} + V_3\vec{k}\right)$

$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ bunu skalerdir.

1-) ∇'nın f'e uygulanışı; grad $f = \vec{\nabla} f \to$ vek.
2-) Div $\vec{v} = \vec{\nabla}\cdot\vec{v} \to$ ska.
3-) Rot $\vec{v} = \vec{\nabla}\wedge\vec{v} \to$ vektör

Diverjansın Özellikleri:

1-) $\vec{\nabla}\circ(\vec{u}+\vec{v}) = \vec{\nabla}\circ\vec{u} + \vec{\nabla}\circ\vec{v}$

2-) f bir skaler bir fonksiyon ise

$\vec{\nabla}\circ(f\vec{u}) = f\,\vec{\nabla}\circ\vec{u} + \vec{u}\,\vec{\nabla} f$

BİR VEKTÖR ALANININ RÖTASYONELİ

$\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$

$\mathrm{rot}\,\vec{V} = \vec{\nabla}\wedge\vec{V}$ $\vec{V}$ vektörünün rötasyonelini almaktır.

Rötasyonelin Özellikleri:

1-) $\vec{\nabla}\wedge\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k}$

2-) $\mathrm{rot}\,(\vec{u}+\vec{v}) = \vec{\nabla}\wedge(\vec{u}+\vec{v}) = \vec{\nabla}\wedge\vec{u} + \vec{\nabla}\wedge\vec{v}$

3-) $\vec{\nabla}\wedge(f\vec{u}) = (f\,\vec{\nabla}\wedge\vec{u}) + (\vec{\nabla} f\wedge\vec{u})$

4-) $\vec{V}$, dönen bir cismin hızı ise rot $\vec{V}$ dönmenin açısal hız vektörünün 2 katını verir. Yani dönme hızını 2 kat artırır.

$f\,\vec{\nabla}\wedge\vec{u} + \vec{\nabla} f\wedge\vec{u}$

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$$\nabla^2 = \nabla \circ \nabla = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \circ \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ Laplasyen

$$\text{Lap } U = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$ u fonksiyonun laplasyeni

$u = f(x, y, z)$ olmak üzere yüzeyin bir D bölgesinde sürekli II. mertebe türevlere malik ise ve laplasyeni 0 ise yani u fonksiyonuna harmonik fonksiyon denir. (Tersi geçerlidir.)

Örnek: $f(x,y,z) = 3x^2y - y^3z^2$ olduğuna göre ∇f $(1,-2,-1)$ noktadaki değerini bulunuz.

$$\nabla f = (6xy)\vec{i} + (3x^2 - 3y^2z^2)\vec{j} + (-2zy^3)\vec{k}$$

$$\nabla f(1,-2,-1) = -12\vec{i} - 9\vec{j} - 16\vec{k}$$

Örnek: $x^2y + 2xz = 4$ yüzeyinin $(2,-2,3)$ noktasındaki birim vektörünü bulunuz.

$$\nabla(x^2y + 2xz - 4) = (2xy + 2z)\vec{i} + (x^2)\vec{j} + (2x)\vec{k}$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$\vec{u} = \frac{\vec{n}}{|\vec{n}|} = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

Örnek: $\vec{F} = (3x^2y - z)\vec{i} + (xz^3 + y^2)\vec{j} + (2x^3z^2)\vec{k}$ olduğuna göre $(2,-1,0)$ noktasında $\nabla(\nabla \circ \vec{F}) = ?$

$$\nabla \circ \vec{F} = 6xy + 2y + 4x^3z$$

$$\nabla(6xy + 2y + 4x^3z) = (6y + 12x^2z)\vec{i} + (6x + 2)\vec{j} + (4x^3)\vec{k}$$

$$\nabla(2,-1,0) = -6\vec{i} + 14\vec{j} + 32\vec{k}$$

Örnek: $\vec{r}$ yer vektörü

$\nabla \wedge \vec{A} = 0$ olduğuna göre $\nabla \circ (\vec{A} \wedge \vec{r}) = 0$ olduğunu gösteriniz.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$ olsun.

$$\vec{A}\wedge\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} = (A_2 z - yA_3)\vec{i} - (A_1 z - A_3 x)\vec{j} + (A_1 y - A_2 x)\vec{k}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$\vec{\nabla}\cdot(\vec{A}\wedge\vec{r}) = 0$$